VI. *On the Diameters of a Plane Cubic.*

By J. J. Walker, *M.A.*, *F.R.S.*

Received June 16,—Read June 16, 1887.
Revised February 9, 1888.

[Plates 6–8.]

I. Abstract.

1. The object of this Memoir is to develop relations which subsist between a cubic (u) and the complex of lines, in its plane, which are the polars with respect to it of the points on any transversal (L). This complex becomes the system of Newtonian Diameters of the cubic, when the points on the plane are projected on a second plane parallel to that containing the vertex of projection and the line L ($lx + my + nz = 0$).

This development involves frequent reference to the envelope of the complex in question, the conic

$$36s = l^2\left\{\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial y\,\partial z}\right)^2\right\} + \ldots + 2mn\left\{\frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y\,\partial z}\right\} + \ldots, \quad . \quad (1)$$

which, in analogy with the "pole" of a line in the theory of conics, I propose to call the "Poloid" of the cubic u and the line L; and, in particular, when the line L is at infinity, the "Centroid" of u.

2. Hesse[*] first appears to have used the equation to s in the theory of the ternary cubic form—but without any recognition of its geometrical significance—to obtain the equation to the cubic in "line-coordinates:" viz., in the form of the resultant of the system

$$\frac{\partial s}{\partial x} : \frac{\partial s}{\partial y} : \frac{\partial s}{\partial z} = l : m : n,$$

with

$$lx + my + nz = 0,$$

and this resultant will plainly be, to a factor, the equation to s itself in line-coordinates ξ, η, ζ, with l, m, n substituted for ξ, η, ζ respectively.

---

[*] 'Crelle, Journ. Math.,' vol. 41, p. 285 . . . , under date January, 1850.

28.6.88

3. In the first edition of the 'Higher Plane Curves' (1852) SALMON, under the subject "Poles and Polars" (of cubics), showed that

$$s = 0$$

is the envelope of the polar lines of points on L, touching each tangent to u at the points $L = 0$, $u = 0$ in the point harmonic conjugate to its contact with u relatively to its intersections with the other two, and proposed for the locus of $s = 0$ the designation "Polar conic of the line" L, for which I have ventured to suggest, and use, the shorter name "Poloid" of (u and) L.

In CAYLEY'S Memoir "On Curves of the Third Order," 'Phil. Trans.,' 1857, the additional property is given of the polar lines of points on L all passing through a point (viz., the intersection of the two right lines into which s, "the lineo-polar envelope of the line," then breaks up) on the Hessian of u, when L joins corresponding points on that curve. Beside these I have not been able to find any notice of the conic s.

4. If through any point P in its plane chords are drawn meeting the cubic in O_1, O_2, O_3, and a point O is taken on the chord determined by the relation

$$\frac{3}{PO} = \frac{1}{PO_1} + \frac{1}{PO_2} + \frac{1}{PO_3},$$

the locus of O is a straight line, according to a theorem of COTES'S, communicated by his friend, Dr. ROBERT SMITH, Master of Trinity College, Cambridge, to MACLAURIN, after COTES'S lamented death, and proved by MACLAURIN* as a case of a more general theorem which presented itself to his mind when "meditating on this communication." For shortness I have called the point O the "COTES-point" on the chords through P, the locus of which is now well known to be the polar line of the point P relative to the cubic u.

5. If P describes a line L, the COTES-points of the polar lines of the points on L—regarded as chords of u—relatively to their intersections with L, may be considered. §§ 28–32.

The locus of the COTES-points of this complex of lines is shown in the sequel to be a nodal cubic (37) and (40),

$$v = 0,$$

which covariant of u and the line L, I propose to call their "*Cotesian.*"

6. Considering (§§ 33–37) more generally the locus of COTES-points on chords of u subject to the condition of touching s, the result comes out as a concomitant breaking up into two factors, one the cubic v just referred to, the other being the equation to

---

* 'De Lin. Geom. Prop. Gen. Theor. IV.,' p. 24, ed. 1748. The Theorem is not given in the 'Harmonia Mensurarum,' as sometimes erroneously stated, with the subjects of which treatise it has no connexion.

the three tangents to u at the points in which it is met by L, now first obtained in a general form. In the Memoir referred to ('Phil. Trans.,' 1857, p. 439) CAYLEY obtained, with his peculiar skill, the equation of the three tangents in question for HESSE's canonical form of u; viz.,

$$ax^3 + by^3 + cz^3 + 6exyz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

finding for the "satellite line" of L, or line in which the tangents again meet u, the equation that, further on (§ 36) will be shown to verify the general form at which I have arrived; and in terms of which (if it be taken as a fundamental covariant of u and L) and of s the equation of the nodal cubic v may be expressed.

7. Whereas through any point in the plane of the cubic u and the line L there can be drawn in general (§§ 40–43) two chords having it as their COTES-point relatively to those in which they meet L, viz., those determined by the polar-conic of the first point; if, however, that point be taken on s the two chords coincide, and thus a *complex of double chords* is obtained, which are the polars with regard to s (§ 42) of the COTES-points of the polar lines of the points of L, and are shown (§§ 58–60) to have as their envelope a tricuspidal quartic (81)

$$w = 0,$$

the equation of which cannot be found explicitly except for specifically assigned forms of u.

The point in which a double chord meets s being its COTES-point, that in which it touches its envelope is shown to be harmonic conjugate to the former with respect to the intersections of the chord with the line L and with the conic s again; and the two intersections of the double chord with s are thereby discriminated (§ 59).

8. Again, considering (§ 61) the points of a line having its pole on L, the chords of which those are COTES-points constitute two groups: viz., one a pencil through the pole of the polar line, the other having as its envelope a conic (§ 88) touching the line L as well as the polar line in question (in virtue of its being the line of the latter complex through its own COTES-point) and the double chord through its point of contact with s, which is both a ray of the pencil and a line of the complex.

This system of conics (§ 62) has as its envelope the sides of the quadrilateral formed by the transversal L and the tangents to the cubic at the points in which L meets it.

9. A question of some interest is considered (§§ 38–39): what, if any, of the complex of polar lines of points on the transversal L are *conjugate*, in the sense of their intersection being the COTES-point on either? Discarding the tangents to u at the points in which it is met by the transversal L, each of which is conjugate to itself, *the only distinct conjugate polar lines of points on* L *are the two tangents to the poloid* (s) *of* L *from the pole of that line with respect to the poloid.*

10. As the chord of contact with the poloid of the two tangents through its pole

L forms a coincident pair of "double chords" (of the cubic); and these three lines, viz., the two tangents and their chord of contact, form a triad of lines of reference by means of which the properties of the complexes of lines here considered may be deduced with far greater facility than through the use of the canonical form (2).

The two tangents to *s* from the pole of L with respect to *s* are the nodal tangents of the Cotesian *v* (§ 50), the line L being its inflexional axis; a circumstance which explains the unique character of this triad of lines, and marks them out as the best system of lines of reference for the discussion of properties connected with this concomitant of the primitive cubic *u* (§§ 48–62).

11. But, whereas these tangents to *s*, the poloid of L, are only real when L cuts *s* in real points (which is shown to occur only when it meets the cubic *u* in but a single real point), three other triads of lines exist, of which one at least is always real, convenient as lines of reference in many of the questions which arise: viz., the sides of one of the three triangles whose corners are the pole of L, with respect to *s*; one of the three points in which L meets *u*; and the pole of the connector of these two points. Each of these being self-conjugate triangles in respect of *s*, the equation of that conic is reduced to a trinomial form; and since the cuspidal tangents of the envelope *w* (§ 76) all pass through the pole of L, the equation of that curve is of a comparatively simple character for one of these triangles of reference. In Plate 6, ABC is one of these triangles (§§ 63–76).

12. If the line L touches *u* it touches its poloid *s* also; and consequently other lines of reference have to be looked for. These are found in the line L, the tangent to *u* at the point where L cuts it (itself a tangent to *s* also, it will be remembered), and the chord of contact of these lines with *s*.

In this case the cubic *v* degenerates into the line L, and a conic having double contact with *s*; while the envelope *w* also degenerates into a conic having double contact with *s* and the Cotesian conic at the same points (§§ 77–81).

13. When L becomes the line at infinity the pencil of chords through any point on it is to be replaced by a system parallel to a given line, and the polar line by the diameter which is the locus of the mean point on any chord of the system relatively to its intersections with the cubic.

The envelope of these diameters I have called the "Centroid" of the cubic, from its evident analogy with the centre of conics, apprehending no confusion in such a connexion with the sense of a "mass-centre," which it sometimes bears.

The consideration of the Centroid and associated curves occupies the concluding part of this Memoir.

14. The method of treatment of the discussions, an abstract of which has just been given, is uniformly analytical, trilinear coordinates being employed. The results will be found to be arrived at without much difficulty, or tedious calculation, considering the great generality of most of them. With a view to simplifying three important discus-

sions as much as possible, a preliminary investigation of a form into which the result of substituting for the variables in a ternary quadratic form the expressions

$$n\frac{\partial\phi}{\partial y} - m\frac{\partial\phi}{\partial z}, \qquad l\frac{\partial\phi}{\partial z} - n\frac{\partial\phi}{\partial x}, \qquad m\frac{\partial\phi}{\partial x} - l\frac{\partial\phi}{\partial y},$$

ϕ being any ternary form and l, m, n any three constants, is introduced (§§ 24–27); and the special cases of its application in the sequel are considered.

15. The notation employed throughout is as follows:—

(i.) The right line, considered generally, is written

$$\xi x + \eta y + \zeta z = 0:$$

(ii.) The particular transversal considered in connexion with a cubic u is always written

$$\mathrm{L} \equiv lx + my + nz:$$

(iii.) The cubic u being, in point coordinates,

$$u \equiv ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + \ldots + 6exyz,$$

is written in line coordinates—or its reciprocal is—

$$v \equiv b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 + \ldots\ldots:$$

(iv.) The poloid of L with respect to the cubic in point coordinates is represented by s as in § 1, or (§ 19)

$$s \equiv u_{11}x^2 + \ldots + u_{23}yz + \ldots\ldots,$$

and in line coordinates—or its reciprocal—by σ, where

$$4\sigma \equiv (4u_{22}u_{33} - u^2{}_{23})\xi^2 + \ldots + 2(u_{31}u_{12} - 2u_{11}u_{23})\eta\zeta + \ldots\ldots:$$

(v.) Any special point on L regarded as the "pole" of a pencil of chords of the cubic u is marked $(x'y'z')$, and any special point on the polar line of $x'y'z'$ is marked $x''y''z''$; a "chord" being a line, or transversal, considered particularly in connexion with its intersections with the cubic u.

II. PRELIMINARY.

16. The intersections of a line $\xi x + \eta y + \zeta z = 0$, with a curve $u = 0$ of order p, may be studied through the equation

$$\rho^p \mathrm{D}^p u + p\rho^{p-1}\mathrm{D}^{p-1}u + \frac{p.p-1}{1.2}.\rho^{p-2}.\mathrm{D}^{p-2}u + \ldots\ldots + p\rho \mathrm{D}u + u = 0, \quad . \quad . \quad (3)$$

where, $x'y'z'$ being a given point on the line, and D defined by

$$p(p-1)\dots(p-r+1)\,\mathrm{D}^r = \left\{(\eta\gamma-\zeta\beta)\frac{\partial}{\partial x'}+(\zeta\alpha-\xi\gamma)\frac{\partial}{\partial y'}+(\xi\beta-\eta\alpha)\frac{\partial}{\partial z'}\right\}^r, \quad (4)$$

if, for shortness, α, β, γ represent the sines of the angles of the triangle of reference, ρ is equal to the length of the segment of the line between the point $x'y'z'$ and any one of the p points in which it meets the curve u ('London Math. Soc. Proc.,' vol. 9, p. 227); provided $\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos\mathrm{A}-2\zeta\xi\cos\mathrm{B}-2\xi\eta\cos\mathrm{C}=1$.

17. Thus the sum of the reciprocals of those segments is equal to

$$-p\,\mathrm{D}u/u\,;$$

while, if (xyz) is any other point on the same line, $\xi x+\dots$, and ρ' similarly equal to the segment between $x'y'z'$ and it,

$$x-x'=\rho'(\eta\gamma-\zeta\beta), \qquad y-y'=\rho'(\zeta\alpha-\xi\gamma), \qquad z-z'=\rho'(\xi\beta-\eta\alpha). \quad . \quad . \quad (5)$$

But if xyz is the COTES-point (§ 4) on the line in respect to $x'y'z'$ and the curve u,

$$\begin{aligned} p/\rho' &= -p\,\mathrm{D}u/u \\ &= -\left\{(\eta\gamma-\zeta\beta)\frac{\partial u'}{\partial x'}+(\zeta\alpha-\xi\gamma)\frac{\partial u'}{\partial y'}+(\xi\beta-\eta\alpha)\frac{\partial u'}{\partial z'}\right\}/u, \end{aligned}$$

whence (5)

$$(x-x')\frac{\partial u'}{\partial x'}+(y-y')\frac{\partial u'}{\partial y'}+(z-z')\frac{\partial u'}{\partial z'}+pu'=0\,;$$

or,

$$x\frac{\partial u'}{\partial x'}+y\frac{\partial u'}{\partial y'}+z\frac{\partial u'}{\partial z'}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

viz., the locus of xyz is the polar line of $x'y'z'$.

18. If now, instead of considering ξ, η, ζ as variable and $(x'y'z')$ as fixed, ξ, η, ζ are regarded as constant—say equal to l, m, n—and $(x'y'z')$ as any point on L, or $lx+my+nz=0$, the envelope of the line (6) will obviously be the same as its equation in line coordinates, regarded as a curve of order $p-1$ in point coordinates $x'y'z'$. Thus, if u were a quartic, (3) having been written in the form

$$x'^3\frac{\partial^3u}{\partial x^3}+\dots\dots+3x'^2y'\frac{\partial^3u}{\partial x^2\,\partial y}+\dots+6x'y'z'\frac{\partial^3u}{\partial x\,\partial y\,\partial z}=0,$$

its envelope might at once be written down as

$$l^6\left\{\left(\frac{\partial^3u'}{\partial y'^3}\right)^2\left(\frac{\partial^3u'}{\partial z'^3}\right)^2+\dots\right\}+m^6\left\{\left(\frac{\partial^3u'}{\partial z'^3}\right)^2\left(\frac{\partial^3u'}{\partial x'^3}\right)^2+\dots\right\}+n^6\left\{\dots\dots\right\}+\dots\dots=0;$$

viz., the poloid of the line L and the quartic u is another quartic. I have made the foregoing remark to draw attention to the perfect generality of the theory of the poloid, and, at the same time, to show that it is of equal or higher degree than u for values of p exceeding 3.

19. In the case, then, alone contemplated in this Memoir, of u being a cubic, and (6) being, therefore, otherwise written

$$x'^2 \frac{\partial^2 u}{\partial x^2} + \ldots\ldots + 2y'z' \frac{\partial^2 u}{\partial y\, \partial z} + \ldots\ldots = 0,$$

the envelope takes the form (1)

$$36\, s \equiv l^2 \left\{ \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial y\, \partial z} \right)^2 \right\} + \ldots\ldots + 2mn \left\{ \frac{\partial^2 u}{\partial z\, \partial x} \frac{\partial^2 u}{\partial x\, \partial y} - \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y\, \partial z} \right\} + \ldots\ldots = 0,$$

as given by SALMON ('Higher Plane Curves,' 3rd edition, p. 156), l, m, n here replacing α, β, γ.

If written in the normal form of a conic the coefficients $u_{11} \ldots u_{23} \ldots$ are the contravariant conics of the triad of conics,

$$\left. \begin{aligned} u_1 &\equiv \frac{\partial u}{\partial x} \Big/ 3, \\ u_2 &\equiv \frac{\partial u}{\partial y} \Big/ 3, \\ u_3 &\equiv \frac{\partial u}{\partial z} \Big/ 3, \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

taken singly and in pairs, viz., the cubic being written as in the 'Higher Plane Curves,' only with the substitution of e for m,

$$u \equiv ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y + 6exyz, \quad (8)$$

and the poloid of u and L being

$$s \equiv u_{11}x^2 + u_{22}y^2 + u_{33}z^2 + u_{22}yz + u_{31}zx + u_{12}xy, \quad \ldots\ldots \quad (9)$$

then

$$\left. \begin{aligned} u_{11} \equiv\, & (b_1c_1 - e^2)l^2 + (ac_1 - a_3{}^2)m^2 + (ab_1 - a_2{}^2)n^2 + 2(a_2a_3 - ae)mn \\ & \qquad + 2(a_2e - a_3b_1)nl + 2(a_3e - a_2c_1)lm \\ u_{22} \equiv\, & (bc_2 - b_3{}^2)l^2 + (a_2c_2 - e^2)m^2 + (ba_2 - b_1{}^2)n^2 + 2(b_1e - a_2b_3)mn \\ & \qquad + 2(b_1b_3 - be)nl + 2(b_3e - b_1c_2)lm \\ u_{33} \equiv\, & (cb_3 - c_2{}^2)l^2 + (ca_3 - c_1{}^2)m^2 + (a_3b_3 - e^2)n^2 + 2(c_1e - c_2a_3)mn \\ & \qquad + 2(c_2e - b_3c_1)nl + 2(c_1c_2 - ce)lm \\ u_{23} \equiv\, & (bc - b_3c_2)l^2 + (ca_2 + c_2a_3 - 2c_1e)m^2 + (ba_3 + a_2b_3 - 2b_1e)n^2 \\ & + 2(b_1c_1 + e^2 - a_2c_2 - a_3b_3)mn + 2(b_1c_2 - bc_1)nl + 2(b_3c_1 - cb_1)lm \\ u_{31} \equiv\, & (cb_1 + b_3c_1 - 2c_2e)l^2 + (ac - a_3c_1)m^2 + (ab_3 + a_3b_1 - 2a_2e)n^2 \\ & + 2(a_2c_1 - ac_2)mn + 2(a_2c_2 + e^2 - b_1c_1 - a_3b_3)nl + 2(c_2a_3 - ca_2)lm \\ u_{12} \equiv\, & (bc_1 + b_1c_2 - 2b_3e)l^2 + (ac_2 + a_2c_1 - 2a_3e)m^2 + (ab - a_2b_1)n^2 \\ & + 2(a_3b_1 - ab_3)mn + 2(b_3a_2 - ba_3)nl + 2(a_3b_3 + e^2 - c_2a_2 - b_1c_1lm) \end{aligned} \right\} \quad (10)$$

20. The invariants of the equation (3), which for $p = 3$, is

$$\rho^3 \mathrm{D}^3 u + 3\rho^2 \mathrm{D}^2 u + 3\rho \,\mathrm{D}u + u = 0, \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

and, for one of the conics, s, $u_1 \ldots$, or $\partial_x u/3 \ldots$,

$$\rho^2 \mathrm{D}^2 u_1 + 2\rho \,\mathrm{D}u_1 + u_1 = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

give the fundamental invariants of L and u, $u_1 \ldots$, in a very succinct form, involving $x'y'z'$ regarded as *parameters* connected by the equation $lx' + my' + nz' = 0$; thus, if

$$\Delta' = \alpha x' + \beta y' + \gamma z',$$

then (11)—ξ, η, ζ being replaced in the operator D (4), § 16, by l, m, n—

$$u_1 \mathrm{D}^2 u_1 - (\mathrm{D}u_1)^2 = \Delta'^2 u_{11} \ldots ; \quad . \; . \; . \; . \; . \; . \; . \quad (13)$$

$$u_2 \mathrm{D}^2 u_3 + u_3 \mathrm{D}^2 u_2 - 2 \,\mathrm{D}u_2 \,\mathrm{D}u_3 = \Delta'^2 u_{23} \ldots ; \quad . \; . \; . \; . \; . \quad (14)$$

('London Math. Soc. Proc.,' vol. 9, p. 232.) Also (1)

$$u \,\mathrm{D}^2 u - (\mathrm{D}u)^2 = \Delta'^2 s, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

$$(u \,\mathrm{D}^3 u - \mathrm{D}u \,\mathrm{D}^2 u)^2 - 4\{(\mathrm{D}u)^2 - u\mathrm{D}^2 u\}\{(\mathrm{D}^2 u)^2 - \mathrm{D}u \,\mathrm{D}^3 u\} = \Delta'^6 v, \quad . \quad (16)$$

if

$$v \equiv b^2 c^2 l^6 + \ldots = 0$$

is the standard form of the condition that L shall touch u.

21. The above forms are very convenient for the comparison of related concomitants essential to the objects of this Memoir.

Thus, the condition that L should touch s (9)

$$(u_{22}u_{33} - u^2_{23}/4)\, l^2 + \ldots + (u_{31}u_{12}/2 - u_{11}u_{23})\, mn + \ldots \;(= 0), \quad . \quad (17)$$

multiplied by Δ'^2 is (if $lx' + my' + nz' = 0$) equal to

$$s \,\mathrm{D}^2 s - (\mathrm{D}s)^2.$$

Now, since $\mathrm{D}^k (\Delta'^{k'}) \equiv 0$, whatever integers k, k' may be,

$$\Delta'^4\{s \,\mathrm{D}^2 s - (\mathrm{D}s)^2\} = \Delta'^2 s 6 \,\mathrm{D}^2 (\Delta'^2 s)^* - 4\{\mathrm{D}(\Delta'^2 s)\}^2 ;$$

---

[*] It is to be observed that if u is of order p, then $\mathrm{D}^r u$ is of order $p - r$; and that if $\phi = \psi\chi$, χ being of order q, ψ of order r, then

$$p \,\mathrm{D}\phi = q\psi \,\mathrm{D}\chi + r\chi \,\mathrm{D}\psi,$$

$$p.p - 1 \,\mathrm{D}^2\phi = q.q - 1\psi \,\mathrm{D}^2\chi + 2qr \,\mathrm{D}\chi \,\mathrm{D}\psi + r.r - 1\chi \,\mathrm{D}^2\psi ;$$

and so on: thus

$$4.3 \,\mathrm{D}^2 (\Delta'^2 s) = 2.1 \,\Delta'^2 \,\mathrm{D}^2 s \ldots$$

or (15)

$$4\,\Delta'^4\{s\,\mathrm{D}^2s-(\mathrm{D}s)^2\}=4\{u\,\mathrm{D}^2u-(\mathrm{D}u^2)\}6\,\mathrm{D}^2\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}\cdot$$
$$-[4\,\mathrm{D}\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}]^2. \quad . \;.\;.\;.\;.\;. \quad (18)$$

But

$$4\,\mathrm{D}\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}=3\,\mathrm{D}u\,\mathrm{D}^2u+u\,\mathrm{D}^3u-4\,\mathrm{D}u\,\mathrm{D}^2u$$
$$=u\,\mathrm{D}^3u-\mathrm{D}u\,\mathrm{D}^2u\,; \quad .\;.\;.\;.\;.\;.\;.\;.\;. \quad (19)$$

and

$$12\,\mathrm{D}^2\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}=3\,\mathrm{D}u\,\mathrm{D}^3u-2\,(\mathrm{D}^2u)^2-\mathrm{D}u\,\mathrm{D}^3u,$$

or

$$6\,\mathrm{D}^2\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}=\mathrm{D}u\,\mathrm{D}^3u-(\mathrm{D}^2u)^2. \quad .\;.\;.\;.\;.\;.\;.\;.\;. \quad (20)$$

By substitution from (19) (20) in (18)

$$4\,\Delta'^4\{s\,\mathrm{D}^2s-(\mathrm{D}s)^2\}=4\{u\,\mathrm{D}^2u-(\mathrm{D}u)^2\}\{\mathrm{D}u\,\mathrm{D}^3u-(\mathrm{D}^2u)^2\}$$
$$-(u\,\mathrm{D}^3u-\mathrm{D}u\,\mathrm{D}^2u)^2,$$

whence (16) (17)

$$(4u_{22}u_{33}-u_{23}{}^2)l^2+\ldots\equiv-(b^2c^2l^6+\ldots), \quad .\;.\;.\;.\;.\;.\;.\;.\;. \quad (21)$$

i.e., the condition that L *shall touch s, is equal to one-fourth of that for* L *touching the cubic u, with changed sign* . . . . (i)

22. Considering next the discriminant of s; viz. (9),

$$8\,\mathrm{D}.^{t}(s)\equiv 2u_{11}(4u_{22}u_{33}-u_{23}{}^2)+u_{12}(u_{23}u_{31}-2u_{33}u_{12})+u_{31}(u_{12}u_{23}-2u_{22}u_{31}) : \quad (22)$$

if, for shortness (7),

$$\left.\begin{array}{lll} v_1=\mathrm{D}u_1, & v_2=\mathrm{D}u_2, & v_3=\mathrm{D}u_3, \\ w_1=\mathrm{D}^2u_1, & w_2=\mathrm{D}^2u_2, & w_3=\mathrm{D}^2u_3, \end{array}\right\}^{*} \quad .\;.\;.\;.\;. \quad (23)$$

$$\left.\begin{array}{lll} \mathrm{U}_1=v_2w_3-v_3w_2, & \mathrm{U}_2=v_3w_1-v_1w_3, & \mathrm{U}_3=v_1w_2-v_2w_1, \\ \mathrm{V}_1=w_2u_3-w_3u_2, & \mathrm{V}_2=w_3u_1-w_1u_3, & \mathrm{V}_3=w_1u_2-w_2u_1, \\ \mathrm{W}_1=u_2v_3-u_3v_2, & \mathrm{W}_2=u_3v_1-u_1v_3, & \mathrm{W}_3=u_1v_2-u_2v_1, \end{array}\right\} \quad . \quad (24)$$

then (13, 14),

$$\Delta'^4(4u_{22}u_{33}-u_{23}{}^2)=4(u_2w_2-v_2{}^2)(u_3w_3-v_3{}^2)-(u_2w_3+u_3w_2-2v_2v_3)^2$$
$$=4(v_2w_3-v_3w_2)(u_2v_3-u_3v_2)-(w_2u_3-w_3u_2)^2$$
$$=4\mathrm{U}_1\mathrm{W}_1-\mathrm{V}_1{}^2\,; \quad .\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (25)$$

. . . .

---

* By means of these expressions the equation of the poloid (s) may be thrown into a form which exhibits it explicitly as the envelope of the polar line of points in L: viz., by (9), (13), (14), (23),

$$\Delta'^2s\equiv(xu'_1+yu'_2+zu'_3)(xw'_1+yw'_2+zw'_3)-(xv'_1+yv'_2+zv'_3)^2. \quad .\;.\;. \quad (8\text{ bis})$$

Of the right lines in this form, (i) $xu'_1+\ldots$ is the polar line of the point $x'y'z'$ in L; (ii) $xw_1+\ldots$ or $(\eta\gamma-\xi\beta)\,v_1+\ldots$ is the Newtonian diameter of chords of u parallel to L (shown in fig. 1 touching the poloid at D′); (iii) $xv'_1+\ldots$ or $x'v_1+\ldots$ is the chord of contact with (s) of (i), (ii).—(April, 1888.)

$$\begin{aligned}\Delta'^4(u_{23}u_{31} - 2u_{33}u_{12}) &= (u_2w_3 + u_3w_2 - 2v_2v_3)\,(u_3w_1 + u_1w_3 - 2v_3v_1)\\ &\quad - 2(u_3w_3 - v_3^2)\,(u_1w_2 + u_2v_1 - 2v_1v_2)\\ &= 2\mathrm{U}_1\mathrm{W}_2 + 2\mathrm{U}_2\mathrm{W}_1 - \mathrm{V}_1\mathrm{V}_2\,. \quad \dots\dots \quad (26)\end{aligned}$$

Hence (22),

$$\begin{aligned}8\Delta'^6\mathrm{D}.^t(s) &= (u_1w_1 + u_1w_1 - 2v_1v_1)\,(2\mathrm{U}_1\mathrm{W}_1 + 2\mathrm{U}_1\mathrm{W}_1 - \mathrm{V}_1\mathrm{V}_1)\\ &\quad + (u_1w_2 + w_2w_1 - 2v_1v_2)\,(2\mathrm{U}_1\mathrm{W}_2 + 2\mathrm{U}_2\mathrm{W}_1 - \mathrm{V}_1\mathrm{V}_2)\\ &\quad + (u_1w_3 + u_3w_1 - 2v_1v_3)\,(2\mathrm{U}_1\mathrm{W}_3 + 2\mathrm{U}_3\mathrm{W}_1 - \mathrm{V}_1\mathrm{V}_3)\\ &= 2\mathrm{U}_1u_1(\mathrm{W}_1w_1 + \mathrm{W}_2w_2 + \mathrm{W}_3w_3) + 2\mathrm{W}_1w_1(\mathrm{U}_1u_1 + \dots)\\ &\quad + 2\mathrm{V}_1v_1(\mathrm{V}_1v_1 + \dots)\\ &= 2\Delta'^6\mathrm{P}^2, \quad \dots\dots \quad (27)\end{aligned}$$

where,

$$\begin{aligned}\mathrm{P} &= (\mathrm{U}_1u_1 + \,.\,.)/\Delta^3 = (\mathrm{V}_1v_1 + \,.\,.)/\Delta^3 = (\mathrm{W}_1w_1 + \,.\,.)/\Delta^3\\ &= (\mathrm{U}_1u_1 + \mathrm{V}_1v_1 + \mathrm{W}_1w_1)/\Delta^3\,;\end{aligned}$$

viz., P is the condition that L should be cut in involution by

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = 0,$$

which condition, multiplied by Δ^3, has been shown ('London Math. Soc. Proc.,' vol. 9, p. 233) to be

$$u_1(\mathrm{D}u_2\mathrm{D}^2u_3 - \mathrm{D}u_3\mathrm{D}^2u_2) + u_2(\mathrm{D}u_3\mathrm{D}^2u_1 - \mathrm{D}u_1\mathrm{D}^2u_3) + u_3(\mathrm{D}u_1\mathrm{D}^2u_2 - \mathrm{D}u_2\mathrm{D}^2u_1) = 0.$$

Thus it is proved (27) that *four times the discriminant of s is equal to the square of the Caleyan of u.* . . . . (ii)

23. If the line $lx + my + nz = 0$ joins corresponding points on the Hessian of u its coefficients satisfy the Cayleyan

$$\mathrm{P} = 0\,;$$

hence its poloid s breaks up into two right lines through the intersection of which the tangents of s all pass; and the locus of this intersection, as the line L varies in position, is the Hessian itself. This is the property mentioned in the Introduction (§ 3) as proved by CAYLEY, 'Phil. Trans.,' 1857, p. 432.

24. A theorem on the result of certain substitutions, which will be of use in subsequent investigations, may be conveniently considered before entering upon them:—

If ϕ is any ternary form of order p, and ψ a quadric—say

$$\psi = \mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}zx + 2\mathrm{h}xy, \quad \dots\dots \quad (28)$$

and if in ψ for xyz are substituted respectively

$$n\frac{\partial\phi}{\partial y}-m\frac{\partial\phi}{\partial z}, \qquad l\frac{\partial\phi}{\partial z}-n\frac{\partial\phi}{\partial x}, \qquad m\frac{\partial\phi}{\partial x}-l\frac{\partial\phi}{\partial y},$$

the result may be thrown, if $\mathrm{L}=lx+my+nz$, into the form

$$\begin{aligned}
&\frac{p\phi}{p-1}\left\{\left(\mathrm{c}\frac{\partial^2\phi}{\partial y^2}+\mathrm{b}\frac{\partial^2\phi}{\partial z^2}-2\mathrm{f}\frac{\partial^2\phi}{\partial y\,\partial z}\right)l^2+\ldots+2\left(-\mathrm{a}\frac{\partial^2\phi}{\partial y\,\partial z}-\mathrm{f}\frac{\partial^2\phi}{\partial x^2}+\mathrm{g}\frac{\partial^2\phi}{\partial x\,\partial y}+\mathrm{h}\frac{\partial^2\phi}{\partial z\,\partial x}\right)mn+\ldots\right\}\\
&-\frac{\mathrm{L}^2}{(p-1)^2}\left\{\mathrm{a}\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z^2}-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\right)^2\right)+\ldots+2\mathrm{f}\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)+\ldots\right\}\\
&+\frac{\mathrm{L}}{(p-1)^2}\left[\left\{\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z^2}-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\right)^2\right)l+\left(\frac{\partial^2\phi}{\partial y\,\partial z}\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right)m+\left(\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z\,\partial x}\right)n\right\}\frac{\partial\psi}{\partial x}\right.\\
&+\qquad\left\{\left(\frac{\partial^2\phi}{\partial y\,\partial z}\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right)l+\left(\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x^2}-\left(\frac{\partial^2\phi}{\partial z\,\partial x}\right)^2\right)m+\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)n\right\}\frac{\partial\psi}{\partial y}\\
&+\qquad\left.\left\{\left(\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z\,\partial x}\right)l+\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)m+\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2\right)n\right\}\frac{\partial\psi}{\partial z}\right]\\
&-\frac{\psi}{(p-1)^2}\left\{\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z^2}-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\right)^2\right)l^2+\ldots+2\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)mn+\ldots\right\}. \qquad (29)
\end{aligned}$$

The proof of this theorem depends on the identities

$$\begin{aligned}
(p-1)^2\left(\frac{\partial\phi}{\partial x}\right)^2&=\left(x\frac{\partial^2\phi}{\partial x^2}+y\frac{\partial^2\phi}{\partial x\,\partial y}+z\frac{\partial^2\phi}{\partial z\,\partial x}\right)^2\\
&=p(p-1)\phi\frac{\partial^2\phi}{\partial x^2}-\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2\right)y^2-\left(\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x^2}-\left(\frac{\partial^2\phi}{\partial z\,\partial x}\right)^2\right)z^2\\
&\qquad+2\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)yz. \qquad . \quad . \quad (30)
\end{aligned}$$

$$\begin{aligned}
(p-1)^2\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}&=\left(x\frac{\partial^2\phi}{\partial x\,\partial y}+y\frac{\partial^2\phi}{\partial y^2}+z\frac{\partial^2\phi}{\partial y\,\partial z}\right)\left(x\frac{\partial^2\phi}{\partial z\,\partial x}+y\frac{\partial^2\phi}{\partial y\,\partial z}+z\frac{\partial^2\phi}{\partial z^2}\right)\\
&=p(p-1)\phi\frac{\partial^2\phi}{\partial y\,\partial z}+\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z^2}-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\right)^2\right)yz+\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)x^2\\
&\qquad-\left(\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z\,\partial x}\right)xy-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right)zx, \qquad . \quad . \quad (31)
\end{aligned}$$

with similar values for

$$\left(\frac{\partial\phi}{\partial y}\right)^2, \qquad \left(\frac{\partial\phi}{\partial z}\right)^2, \qquad \frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}, \qquad \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}.$$

But (28)

$$\psi\left(n\frac{\partial\phi}{\partial y}-m\frac{\partial\phi}{\partial z},\quad l\frac{\partial\phi}{\partial z}-m\frac{\partial\phi}{\partial x},\quad m\frac{\partial\phi}{\partial x}-l\frac{\partial\phi}{\partial y}\right)$$

$$=(\mathrm{c}m^2+\mathrm{b}n^2-2\mathrm{f}mn)\left(\frac{\partial\phi}{\partial x}\right)^2+(\mathrm{a}n^2+\mathrm{c}l^2-2\mathrm{g}nl)\left(\frac{\partial\phi}{\partial y}\right)^2+(\mathrm{b}l^2+\mathrm{a}m^2-2\mathrm{h}lm)\left(\frac{\partial\phi}{\partial z}\right)^2$$

$$+2(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}+2(-\mathrm{b}nl+\mathrm{f}lm-\mathrm{g}m^2+\mathrm{h}mn)\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}$$

$$+2(-\mathrm{c}lm+\mathrm{f}nl+\mathrm{g}mn-\mathrm{h}n^2)\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y};$$

the substitution of the values (30), (31) of $(\partial\phi/\partial x)^2, \ldots, \partial\phi/\partial y\ \partial\phi/\partial z, \ldots$, in which, with the addition and subtraction of the terms, wherein $\mathrm{L} = lx + my + nz$,

$$2lx\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z^2}-\left(\frac{\partial^2\phi}{\partial y\,\partial z}\right)^2\right)(\mathrm{La}+l\mathrm{h}y+l\mathrm{g}z),$$

$$2my\left(\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x^2}-\left(\frac{\partial^2\phi}{\partial z\,\partial x}\right)^2\right)(m\mathrm{h}x+\mathrm{Lb}+m\mathrm{f}z),$$

$$2nz\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2\right)(n\mathrm{g}x+n\mathrm{f}y+\mathrm{Lc}),$$

$$2\left(\frac{\partial^2\phi}{\partial z\,\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y\,\partial z}\right)\{mn(\mathrm{b}y^2+\mathrm{c}z^2+\mathrm{g}zx+\mathrm{h}xy)+\mathrm{Lf}(my+nz)\},$$

$$2\left(\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial z\,\partial x}\right)\{nl(\mathrm{c}z^2+\mathrm{a}x^2+\mathrm{h}xy+\mathrm{f}yz)+\mathrm{Lg}(nz+lx)\},$$

$$2\left(\frac{\partial^2\phi}{\partial y\,\partial z}\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial z^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right)\{lm(\mathrm{a}x^2+\mathrm{b}y^2+\mathrm{f}yz+\mathrm{g}zx)+\mathrm{Lh}(lx+my)\},$$

gives an expression identically equal to the form (29).

25. The cases of the application of this theorem which occur are :—(1) When (§ 28) ϕ is a cubic u, and ψ is the polar line of a point whose coordinates are $n\,\partial u'/\partial y' - m\,\partial u'/\partial z' \ldots$, $(x'y'z')$ being also a point on $lx + my + nz = 0$; so that $\mathrm{a} = \partial^2\phi/\partial x^2 \ldots \mathrm{f} = \partial^2\phi/\partial y\,\partial z \ldots$; in which case

$$\frac{p\phi}{p-1}=\frac{3u'}{2},\qquad \frac{\psi}{(p-1)^2}=\tfrac{1}{4}\left(x'^2\frac{\partial^2 u}{\partial x^2}+\ldots\right)=\tfrac{1}{2}\left(x\frac{\partial u'}{\partial x'}+y\frac{\partial u'}{\partial y'}+z\frac{\partial u'}{\partial z'}\right),$$

and the quadratic functions in l, m, n which multiply ϕ and ψ respectively become

$$(\mathrm{bc}'+\mathrm{b}'\mathrm{c}-2\mathrm{ff}')l^2+\ldots,\qquad +2(\mathrm{gh}'+\mathrm{g}'\mathrm{h}-\mathrm{af}'-\mathrm{a}'\mathrm{f})mn+\ldots,\quad \text{and}\quad 36s'$$

rpesectively, where $\mathrm{a}' = \partial^2u'/\partial x'^2 \ldots \mathrm{f}' = \partial^2u'/\partial y'\,\partial z' \ldots$.

Now, otherwise, the substitutions may be made in the conics $(\partial u/\partial x) \ldots$ of the other form of the polar line,

$$2\left\{x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}\right)+z\left(\frac{\partial u}{\partial z}\right)\right\},$$

and then the multiplier of $3u'/2$ (the present value of $p\phi/(p-1)$ in (29)) becomes, for $\psi = 2\partial u/\partial x$,

$$\left(\frac{\partial^3 u}{\partial x\,\partial z^2}\,\frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^3 u}{\partial x\,\partial y^2}\,\frac{\partial^2 u'}{\partial z'^2} - 2\,\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}\,\frac{\partial^2 u'}{\partial y'\,\partial z'}\right) l^2 + \ldots ;$$

or, identically, that multiplier becomes

$$36\,\frac{\partial s'}{\partial x'}, \qquad 36\,\frac{\partial s'}{\partial y'}, \qquad 36\,\frac{\partial s'}{\partial z'}$$

for

$$\psi = 2\,\frac{\partial u}{\partial x}, \qquad \psi = 2\,\frac{\partial u}{\partial y}, \qquad \psi = 2\,\frac{\partial u}{\partial z}$$

respectively.

Thus the result of the substitutions becomes

$$18\left\{u'\left(x\,\frac{\partial s'}{\partial x'} + y\,\frac{\partial s'}{\partial y'} + z\,\frac{\partial s'}{\partial z'}\right) - \left(x\,\frac{\partial u'}{\partial x'} + y\,\frac{\partial u'}{\partial y'} + z\,\frac{\partial u'}{\partial z'}\right)s'\right\}; \quad . \quad . \quad . \quad (32)$$

whereas, by making the substitutions in the first way, the result would be

$$\frac{3u'}{2}\left\{(\mathrm{bc}' + \mathrm{b}'\mathrm{c} - 2\mathrm{ff}')\,l^2 + \ldots\right\} - 18s'\left(x\,\frac{\partial u'}{\partial x'} + \ldots\right).$$

Identifying these two results,

$$(\mathrm{bc}' + \mathrm{b}'\mathrm{c} - 2\mathrm{ff}')\,l^2 + \ldots + 2\,(\mathrm{gh}' + \mathrm{g}'\mathrm{h} - \mathrm{af}' - \mathrm{a}'\mathrm{f})\,mn + \ldots$$
$$\equiv 12\left(x\,\frac{\partial s'}{\partial x'} + y\,\frac{\partial s'}{\partial y'} + z\,\frac{\partial s'}{\partial z'}\right);$$

so that the polar of $x'y'z'$ with respect to the poloid of L may be written

$$\left(\frac{\partial^2 u'}{\partial z'^2}\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u'}{\partial y'^2}\,\frac{\partial^2 u}{\partial z^2} - 2\,\frac{\partial^2 u'}{\partial y'\,\partial z'}\,\frac{\partial^2 u}{\partial y\,\partial z}\right) l^2 + \ldots + 2\left(\frac{\partial^2 u'}{\partial x'\,\partial y'}\,\frac{\partial^2 u}{\partial z\,\partial x} + \frac{\partial^2 u'}{\partial z'\,\partial x'}\,\frac{\partial^2 u}{\partial x\,\partial y}\right.$$
$$\left. - \frac{\partial^2 u'}{\partial y'\,\partial z'}\,\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u'}{\partial x'^2}\,\frac{\partial^2 u}{\partial y\,\partial z}\right) mn + \ldots = 0. \quad (33)$$

26. In the second case which occurs of similar substitutions (§ 30)

$$p\,\phi\,/\,(p-1) = 2s$$
$$\psi\,/\,(p-1)^2 = x^2\,\frac{\partial^2 u}{\partial x^2} + \ldots + 2yz\,\frac{\partial^2 u}{\partial y\,\partial x}$$
$$= 6u$$
$$\frac{\partial^2 \phi}{\partial x^2} \ldots \frac{\partial^2 \phi}{\partial y\,\partial z} \ldots = 2u_{11} \ldots u_{23} \ldots$$

the doubles of the coefficients of s; and

$$lx + my + nz = \mathrm{L}$$

no longer vanishes. The result (29), § 24, thus becomes

$$\begin{aligned}2s\,\Sigma\Big\{&\Big(2u_{33}\frac{\partial^2u}{\partial y^2}+2u_{22}\frac{\partial^2u}{\partial z^2}-2u_{23}\frac{\partial^2u}{\partial y\,\partial z}\Big)l^2+2\Big(-2u_{11}\frac{\partial^2u}{\partial y\,\partial z}-u_{23}\frac{\partial^2u}{\partial x^2}+u_{31}\frac{\partial^2u}{\partial x\,\partial y}+u_{12}\frac{\partial^2u}{\partial z\,\partial x}\Big)mn\Big\}\\ &-\mathrm{L}^2\Sigma\Big\{(4u_{22}u_{33}-u_{23}{}^2)\frac{\partial^2u}{\partial x^2}+2(u_{31}u_{12}-2u_{11}u_{23})\frac{\partial^2u}{\partial y\,\partial z}\Big\}\\ &+4\mathrm{L}\Sigma\,\{(4u_{22}u_{33}-u_{23}{}^2)\,l+(u_{23}u_{31}-2u_{33}u_{12})\,m+(u_{12}u_{23}-2u_{22}u_{31})\,n\,\}\,\frac{\partial u}{\partial x}\\ &-6u\Sigma\,\{(4u_{22}u_{33}-u_{23}{}^2)\,l^2+2(u_{31}u_{12}-2u_{11}u_{23})\,mn\}. \qquad \text{. . . . .} \qquad (34)\end{aligned}$$

It is to be observed that the values of $\partial\psi/\partial x$ . . . are to be obtained by substituting $\partial^2u/\partial x^2$ . . . for a . . . *after* differentiation; thus, generally, in the present case

$$\begin{aligned}\frac{\partial\psi}{\partial x}&=2\mathrm{a}x+2\mathrm{h}y+2\mathrm{g}z\\ &=2\Big(x\frac{\partial^2u}{\partial x^2}+y\frac{\partial^2u}{\partial x\,\partial y}+z\frac{\partial^2u}{\partial z\,\partial x}\Big),\text{ or }4\frac{\partial u}{\partial x}.\end{aligned}$$

27. Lastly, the case which will occur at § 38 of the substitutions,

$$n\frac{\partial u'}{\partial y'}-m\frac{\partial u'}{\partial z'}\ \ldots \text{ for } xyz \text{ in } x'\frac{\partial u}{\partial x}+y'\frac{\partial u}{\partial y}+z'\frac{\partial u}{\partial z},$$

differs from the first case only in the differential coefficients of u being multiplied by x', y', z' instead of x, y, z, so that (29) becomes simply equal to $-u's'$; *i.e.*, $-us$ with $x'y'z'$ for xyz.

III. General Formulæ and Equations.

28. The polar line of a point $(x'y'z')$ on L, or $lx+my+nz=0$, meets it in a second point, the coordinates of which are

$$\begin{aligned}x\quad &:\quad y\quad :\quad z\\ =n\frac{\partial u'}{\partial y'}-m\frac{\partial u'}{\partial z'}\ &:\ l\frac{\partial u'}{\partial z'}-n\frac{\partial u'}{\partial x'}\ :\ m\frac{\partial u'}{\partial x'}-l\frac{\partial u'}{\partial y'}\,;\end{aligned}$$

and if the coordinates of this latter point are substituted for xyz in the quadric forms $(\partial u/\partial x)$, $(\partial u/\partial y)$, $(\partial u/\partial z)$ of

$$x\Big(\frac{\partial u}{\partial x}\Big)+y\Big(\frac{\partial u}{\partial y}\Big)+z\Big(\frac{\partial u}{\partial z}\Big)=0,$$

the polar line of this point will be expressed in terms of the coordinates of the original point $(x'y'z')$.

Referring to the general form for the result of such substitutions, (29), § 24 in the present case it is (32), § 25,

$$u'\left(x\frac{\partial s'}{\partial x'}+y\frac{\partial s'}{\partial y'}+z\frac{\partial s'}{\partial z'}\right)-s'\left(x\frac{\partial u'}{\partial x'}+y\frac{\partial u'}{\partial y'}+z\frac{\partial u'}{\partial z'}\right), \quad . \quad . \quad . \quad . \quad (35)$$

the form of which shows that *the polar with respect to the poloid* (s) *of any point on* L *cuts the polar line with respect to* u *of that point in its* COTES-*point.*

29. The relation just proved gives at once the coordinates of that COTES-point in the form

$$\frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial y'}-\frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial z'}:\frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial z'}-\frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial x'}:\frac{\partial s'}{\partial y'}\frac{\partial s'}{\partial x'}-\frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial y'}. \quad . \quad . \quad . \quad . \quad (36)$$

And, plainly, *the* COTES-*point of the polar line of a point on* L *is the pole* (with respect to (s) the poloid of L) *of that chord of the pencil through the point on* L, *which passes through the contact with* s *of the polar line of that point.* (Fig. 1.)

30. From considerations founded on the relation just established the locus of the COTES-point (xyz) of the polar line of a point ($x'y'z'$) on L may be at once obtained in a general form, viz., by the elimination of $x'y'z'$ among

$$x\frac{\partial u'}{\partial x'}+y\frac{\partial u'}{\partial y'}+z\frac{\partial u'}{\partial z'},$$

or

$$x'^2\frac{\partial^2 u}{\partial x^2}+y'^2\frac{\partial^2 u}{\partial y^2}+z'^2\frac{\partial^2 u}{\partial z^2}+2y'z'\frac{\partial^2 u}{\partial y\,\partial z}+2z'x'\frac{\partial^2 u}{\partial z\,\partial x}+2x'y'\frac{\partial^2 u}{\partial x\,\partial y}=0,$$

$$x\frac{\partial s'}{\partial x'}+y\frac{\partial s'}{\partial y'}+z\frac{\partial s'}{\partial z'},$$

or

$$x'\frac{\partial s}{\partial x}+y'\frac{\partial s}{\partial y}+z'\frac{\partial s}{\partial z}=0,$$

$$lx'+my'+nz'=0.$$

From the last two

$$x' \quad : \quad y' \quad : \quad z'$$

$$=n\frac{\partial s}{\partial y}-m\frac{\partial s}{\partial z}:l\frac{\partial s}{\partial z}-n\frac{\partial s}{\partial x}:m\frac{\partial s}{\partial x}-l\frac{\partial s}{\partial y},$$

and the substitution of these values in the first gives the locus of xyz in the form (34) § 26, of six times v, if

$$
\begin{aligned}
v \equiv & -4u\,(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm) \\
& - \frac{2L^2}{3}\left\{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y^2} + C\frac{\partial^2 u}{\partial z^2} + 2F\frac{\partial^2 u}{\partial y\,\partial z} + 2G\frac{\partial^2 u}{\partial z\,\partial x} + 2H\frac{\partial^2 u}{\partial x\,\partial y}\right\} \\
& + \frac{8L}{3}\left\{(Al + Hm + Gn)\frac{\partial u}{\partial x} + (Hl + Bm + Fn)\frac{\partial u}{\partial y} + (Gl + Fm + Cn)\frac{\partial u}{\partial z}\right\} \\
& + \frac{2s}{3}\left\{\left(u_{33}\frac{\partial^2 u}{\partial y^2} + u_{22}\frac{\partial^2 u}{\partial z^2} - u_{23}\frac{\partial^2 u}{\partial y\,\partial z}\right)l^2 + \ldots\right. \\
& \qquad \left. + \left(-2u_{11}\frac{\partial^2 u}{\partial y\,\partial z} - u_{23}\frac{\partial^2 u}{\partial x^2} + u_{31}\frac{\partial^2 u}{\partial x\,\partial y} + u_{12}\frac{\partial^2 u}{\partial z\,\partial x}\right)mn \ldots\right\} = 0, \quad . \quad (37)
\end{aligned}
$$

where, as above (7), $u_{11}\ldots, u_{23}\ldots$ are the contravariants of

$$u_1 = \tfrac{1}{3}\frac{\partial u}{\partial x}, \qquad u_2 = \tfrac{1}{3}\frac{\partial u}{\partial y}, \qquad u_3 = \tfrac{1}{3}\frac{\partial u}{\partial z},$$

taken singly and in pairs, these being the coefficients of the poloid, viz.,

$$s \equiv u_{11}x^2 + u_{22}y^2 + u_{33}z^2 + u_{23}yz + u_{31}zx + u_{12}xy\,;$$

and A, B, C, F, G, H are the coefficients of its reciprocal, viz.,

$$
\begin{aligned}
4A &\equiv 4u_{22}u_{33} - u_{23}{}^2, \ldots \\
4F &\equiv u_{31}u_{12} - 2u_{11}u_{23}, \ldots
\end{aligned}
$$

31. The coefficient of s in the above expression admits of an important transformation, viz., it may be written

$$4\left\{u_{11}\left(n\frac{\partial}{\partial y} - m\frac{\partial}{\partial z}\right)^2 + \ldots + u_{23}\left(l\frac{\partial}{\partial z} - n\frac{\partial}{\partial x}\right)\left(m\frac{\partial}{\partial x} - l\frac{\partial}{\partial y}\right) + \ldots\right\}u/6,$$

and this (71), § 55, is identically equal to

$$-4P\,(lx + my + nz), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where P is the Pippian, or Cayleyan, of the cubic u.

Also (55)

$$
\begin{aligned}
& 16\left\{(Al + Hm + Gn)\frac{\partial u}{\partial x} + (Hl + Bm + Fn)\frac{\partial u}{\partial y} + (Gl + Fm + Cn)\frac{\partial u}{\partial z}\right\} + 24Ps \\
& \equiv -\left\{(A' + 12A)\frac{\partial^2 u}{\partial x^2} + \ldots + 2\,(F' + 12F)\frac{\partial^2 u}{\partial y\,\partial z} + \ldots\right\}(lx + my + nz), \quad . \quad (39)
\end{aligned}
$$

$6\mathrm{A}', \ldots 6\mathrm{F}', \ldots$ being the second differential coefficients with respect to l, m, n of the reciprocal (υ) of u, $b^2c^2l^6 \ldots$; while, otherwise, (21) § 21,

$$\upsilon \equiv -4\,(\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2 + 2\mathrm{F}mn + 2\mathrm{G}nl + 2\mathrm{H}lm).$$

With these substitutions the equation to the locus of the COTES-points on the polar lines of points in L, or $lx + my + nz$, takes the form

$$\boldsymbol{\upsilon} \equiv \upsilon u - \frac{\mathrm{L}^2}{6}\left\{(\mathrm{A}' + 16\mathrm{A})\frac{\partial^2 u}{\partial x^2} + \ldots + 2\,(\mathrm{F}' + 16\mathrm{F})\frac{\partial^2 u}{\partial y\,\partial z} + \ldots\right\} - 8\mathrm{PL}s = 0, \quad . \quad (40)$$

a cubic, degenerating into a conic when the line L touches the cubic u, which will be traced further on (§§ 50–53) by referring it to the line L and the two tangents to s at the points $\mathrm{L} = 0$, $s = 0$. But previously it will be of interest to show the significance of the part

$$\upsilon u - \mathrm{L}^2\left\{(\mathrm{A}' + 16\mathrm{A})\frac{\partial^2 u}{\partial x^2} + \ldots + 2\,(\mathrm{F}' + 16\mathrm{F})\frac{\partial^2 u}{\partial y\,\partial z} + \ldots\right\}/6 \quad . \quad . \quad (41)$$

in the equation to $\boldsymbol{\upsilon}$; and to add a few remarks on the relations (38) (39).

32. It is not easy to verify these relations by means of the invariants of (12) § 20, because the variables which enter into them are perfectly general—except satisfying $\alpha x + \beta y + \gamma z =$ constant. They are verifiable, the former with slight, the latter with moderate labour by means of the canonical form of the equation to the cubic; but much more readily by means of the simultaneous forms of u and s, to which every form of u and its concomitant are reducible, referred to in § 10. The verification is therefore deferred until the reduction to those forms is explained in the sequel, and will be found in the paragraphs cited.

33. The tangent to u at any one of the three points in which it is met by the transversal L, being the polar line of that point, also touches the poloid s, the point of contact with s being its COTES-point—viz., it is that determined on it by the polar with respect to s of the point in which it meets L; or, otherwise, as the point in which it is cut by the coincident tangent.

The general equation to the three tangents, at the points $u = 0$, $\mathrm{L} = 0$, may be obtained without the difficulties which would attend the direct investigation—hitherto unattempted, at least successfully—through the property of their touching the poloid also, as follows: the point of contact of one of the tangents being $(x'y'z')$, its equation is

$$x\frac{\partial u'}{\partial x'} + y\frac{\partial u'}{\partial y'} + z\frac{\partial u'}{\partial z'} = 0,$$

or, u being a cubic,

$$x'^2\frac{\partial^2 u}{\partial x^2} + y'^2\frac{\partial^2 u}{\partial y^2} + z'^2\frac{\partial^2 u}{\partial z^2} + 2y'z'\frac{\partial^2 u}{\partial y\,\partial z} + 2z'x'\frac{\partial^2 u}{\partial z\,\partial y} + 2x'y'\frac{\partial^2 u}{\partial x\,\partial y} = 0. \quad . \quad (42)$$

But in virtue of its touching s also, if (xyz) is any other point on it,

$$yz' - zy', \qquad zx' - xz', \qquad xy' - yx'$$

must satisfy the reciprocal (§ 30) or tangential of s,

$$\mathrm{A}\xi^2 + \mathrm{B}\eta^2 + \mathrm{C}\zeta^2 + 2\mathrm{F}\eta\zeta + 2\mathrm{G}\zeta\xi + 2\mathrm{H}\xi\eta = 0\,;$$

i.e.,

$$\begin{aligned}(\mathrm{C}y^2 + \mathrm{B}z^2 - 2\mathrm{F}yz)\,x'^2 + (\mathrm{A}z^2 + \mathrm{C}x^2 - 2\mathrm{G}zx)\,y'^2 + (\mathrm{B}x^2 + \mathrm{A}y^2 - 2\mathrm{H}xy)z'^2 \\ + 2\,(-\mathrm{A}yz - \mathrm{F}x^2 + \mathrm{G}xy + \mathrm{H}zx)\,y'z' + 2\,(-\mathrm{B}zx + \mathrm{F}xy - \mathrm{G}y^2 + \mathrm{H}yz)\,z'x' \\ + 2\,(-\mathrm{C}xy + \mathrm{F}zx + \mathrm{G}yz - \mathrm{H}z^2)x'y' = 0. \quad \ldots\ldots\ldots \quad (43)\end{aligned}$$

The eliminant of these two quadrics (42, 43) with

$$lx' + my' + nz' = 0$$

in the known form

$$\begin{aligned}\{(\mathrm{bc}' + \mathrm{b}'\mathrm{c} - 2\mathrm{f}\mathrm{f}')l^2 + \ldots + 2\,(\mathrm{gh}' + \mathrm{g}'\mathrm{h} - \mathrm{af}' - \mathrm{a}'\mathrm{f})\,mn + \ldots\}^2 \\ - 4\,\{(\mathrm{bc} - \mathrm{f}^2)\,l^2 + \ldots + 2\,(\mathrm{gh} - \mathrm{af})\,mn + \ldots\}\{(\mathrm{b}'\mathrm{c}' - \mathrm{f}'^2)\,l^2 + \ldots\} = 0, \quad (44)\end{aligned}$$

is, if

$$\mathrm{L} = lx + my + nz,$$

$$(v + 4\mathrm{PL}s)^2 - 16\mathrm{P}^2\mathrm{L}^2s^2.$$

34. For

$$(\mathrm{bc} - \mathrm{f}^2)\,l^2 + \ldots \qquad + 2\,(\mathrm{gh} - \mathrm{af})\,mn + \ldots \qquad \equiv 36s,$$

$$\begin{aligned}4\,\{(\mathrm{b}'\mathrm{c}' - \mathrm{f}'^2)\,l^2 + \ldots\} &\equiv 4\,\{(\mathrm{BC} - \mathrm{F}^2)\,x^2 + \ldots + 2\,(\mathrm{GH} - \mathrm{AF})\,yz\}(lx + my + nz)^2 \\ &\equiv 4\,(\text{Discrt. of } s)\,\mathrm{L}^2s \\ &\equiv \mathrm{P}^2\mathrm{L}^2s\,(27).\end{aligned}$$

Hence

$$4\,\{(\mathrm{bc} - \mathrm{f}^2)\,l^2 + \ldots\}\{(\mathrm{b}'\mathrm{c}' - \mathrm{f}'^2)\,l^2 + \ldots\} \equiv 36\mathrm{P}^2\mathrm{L}^2s^2. \quad \ldots \quad (45)$$

Observing now that

$$4\,\mathrm{A} \equiv \frac{\partial^2 s}{\partial y^2}\frac{\partial^2 s}{\partial z^2} - \left(\frac{\partial^2 s}{\partial y\,\partial z}\right)^2 \ldots 4\mathrm{F} = \frac{\partial^2 s}{\partial z\,\partial x}\frac{\partial^2 s}{\partial x\,\partial y} - \frac{\partial^2 s}{\partial x^2}\frac{\partial^2 s}{\partial y\,\partial z}\ldots,$$

it appears at once from (30, 31) § 24 that—substituting s for ϕ—

$$4\,(\mathrm{C}y^2 + \mathrm{B}z^2 - \mathrm{F}yz) = 4u_{11}s - \left(\frac{\partial s}{\partial x}\right)^2 \ldots$$

$$4\,(-\mathrm{A}yz - \mathrm{F}x^2 + \mathrm{G}xy + \mathrm{H}zx) = 2u_{23}s - \frac{\partial s}{\partial y}\frac{\partial s}{\partial z}\ldots,$$

so that

$$4\;(\mathrm{bc}' + \mathrm{b}'\mathrm{c} - 2\mathrm{ff}')\,l^2 + \ldots$$

$$= 4s\left\{\left(u_{33}\frac{\partial^2 u}{\partial y^2} + u_{22}\frac{\partial^2 u}{\partial z^2} - u_{23}\frac{\partial^2 u}{\partial y\,\partial z}\right)l^2 + \ldots\right.$$

$$\left. + \left(-2u_{11}\frac{\partial^2 u}{\partial y\,\partial z} - u_{23}\frac{\partial^2 u}{\partial x} + u_{31}\frac{\partial^2 u}{\partial x\,\partial y} - u_{12}\frac{\partial^2 u}{\partial z\,\partial x}\right)mn + \ldots\right\}$$

$$- \left\{\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial s}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial s}{\partial z}\right)^2 - 2\,\frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial s}{\partial y}\frac{\partial s}{\partial z}\right\}l^2 - \ldots$$

$$- 2\left\{-\frac{\partial^2 u}{\partial x^2}\frac{\partial s}{\partial y}\frac{\partial s}{\partial z} - \frac{\partial^2 u}{\partial y\,\partial z}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial s}{\partial x}\frac{\partial s}{\partial y} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial s}{\partial z}\frac{\partial s}{\partial x}\right\}mn - \ldots$$

$$= 4s\{\ldots\} - \left\{\frac{\partial^2 u}{\partial x^2}\left(n\frac{\partial s}{\partial y} - m\frac{\partial s}{\partial z}\right)^2 + \ldots + 2\frac{\partial^2 u}{\partial y\,\partial z}\left(l\frac{\partial s}{\partial z} - n\frac{\partial s}{\partial x}\right)\left(m\frac{\partial s}{\partial x} - l\frac{\partial s}{\partial y}\right) + \ldots\right\},$$

the negative term in which being the result of substituting $n\,\partial s/\partial y - m\,\partial s/\partial z$ for $x'\ldots$ in

$$x'^2\frac{\partial^2 u}{\partial x^2} + \ldots + 2y'z'\frac{\partial^2 u}{\partial y\,\partial z} + \ldots,$$

is (§ 30)

$$-6v; \quad \ldots\ldots\ldots\ldots \quad (46)$$

consequently, (45) (46), sixteen times the eliminant (44) is

$$\left[-6v + 4s\left\{u_{11}\left(n\frac{\partial}{\partial y} - m\frac{\partial}{\partial z}\right)^2 + \ldots + u_{23}\left(l\frac{\partial}{\partial z} - n\frac{\partial}{\partial x}\right)\left(m\frac{\partial}{\partial x} - l\frac{\partial}{\partial y}\right) + \ldots\right\}u\right]^2$$

$$- 576\mathrm{P}^2\mathrm{L}^2s^2 = 0. \quad \ldots\ldots \quad (47)$$

As was mentioned, (38), § 31, the terms multiplied by $4s$ within [ ]^2 in (47) are equal to

$$-6\mathrm{PL},$$

so that the eliminant is simply, after division by 36,

$$(v + 4\mathrm{PL}s)^2 - 16\mathrm{P}^2\mathrm{L}^2s^2 = 0$$

or

$$v\,(v + 8\mathrm{PL}s) = 0.$$

35. The presence of the factor v in this result is accounted for by the fact that the solution has been really that of the more general question: "to find the locus of the Cotes-points of all chords of u (relatively to their intersections with L) which touch s"; discarding, therefore, this factor, the equation of the tangents to u at the points

$$u = 0, \qquad \mathrm{L} \equiv lx + my + nz = 0,$$

is

$$\varpi \equiv v + 8\mathrm{PL}s = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Reverting now to the form (40) of v, in which

$$\mathrm{A}' = \frac{\partial^2 v}{\partial \xi^2}/6 \ldots\ldots \mathrm{A} = \frac{\partial^2 \sigma}{\partial \xi^2}/2 \ldots\ldots,$$

the equation of the three tangents to u at the points

$$u = 0, \qquad lx + my + nz = 0$$

is found in its standard form; viz.,

$$\varpi \equiv vu - \mathrm{L}^2\Sigma \left\{ \left(\frac{\partial^2 v}{\partial \xi^2} + 48\,\frac{\partial^2 \sigma}{\partial \xi^2}\right)\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial^2 v}{\partial \eta\,\partial \zeta} + 48\,\frac{\partial^2 \sigma}{\partial \eta\,\partial \zeta}\right)\frac{\partial^2 u}{\partial y\,\partial z}\right\}/36, \quad . \quad (49)$$

wherein ξ, η, ζ are to be replaced in v and its second differential coefficients by l, m, n.

36. It will be satisfactory to verify the general expression for the satellite chord of L by applying it to the canonical form of u, for which Cayley has obtained the form referred to in § 6.

For

$$u \equiv ax^3 + by^3 + cz^3 + 6exyz,$$

$$\tfrac{1}{6}\frac{\partial^2 v}{\partial \xi^2} = 5b^2c^2l^4 - 2(abc + 16e^3)(cm^3l + bn^3l) - 48bce^2l^2mn - 8(abc + 2e^3)em^2n^2,$$

$$8\,\frac{\partial^2 \sigma}{\partial \xi^2} = -\,4b^2c^2l^4 + 32e^3(cm^3l + bn^3l) + 48bce^2lmn,$$

$$\tfrac{1}{6}\frac{\partial^2 u}{\partial x^2} = ax\,;$$

whence,

$$\tfrac{1}{36}\left(\frac{\partial^2 v}{\partial \xi^2} + 48\,\frac{\partial^2 \sigma}{\partial \xi^2}\right)\frac{\partial^2 u}{\partial x^2} = \{abc(bcl^4 - 2cam^3l - 2abn^3l) - 8(abc + 2e^3)aem^2n^2\}x\ldots \quad (50)$$

Again,

$$\tfrac{1}{3}\frac{\partial^2 v}{\partial\eta\,\partial\zeta} = -\,8e^2(bcl^4 + 4cam^3l + 4abn^3l) - 32(abc + 2e^3)el^2mn - 6(abc + 16e^3)am^2n^2,$$

$$16\,\frac{\partial^2\sigma}{\partial\eta\,\partial\zeta} = 16e^2(bcl^4 + cam^3l + abn^3l) + 32(abc + 2e^3)el^2mn + 8(abc + 8e^3)am^2n^2,$$

$$\tfrac{1}{6}\frac{\partial^2 u}{\partial y\,\partial z} = ex\,;$$

whence,

$$\tfrac{2}{36}\left(\frac{\partial^2 v}{\partial\eta\,\partial\zeta} + 48\frac{\partial^2\sigma}{\partial\eta\,\partial\zeta}\right)\frac{\partial^2 u}{\partial y\,\partial z} = \{8e^3(bcl^4 - 2cam^3l - 2abn^3l) + 2(abc - 16e^3)aem^2n^2\}x \ldots \quad (51)$$

From (50) (51) there results, as the equation of the satellite chord of L,

$$(abc + 8e^3)\,\Sigma\,(bcl^4 - cam^3l - 2abn^3l - 6aem^2n^2)\,x\,; \quad . \quad . \quad . \quad . \quad (52)$$

which, if $a = b = c = 1$, and e, l, m, n be replaced by m, α, β, γ respectively, agrees exactly with the form given in the "Memoir on Curves of the Third Order" ('Phil. Trans.,' 1857, p. 439).

37. The values of the second differential coefficients of σ and v used in the preceding verification may be calculated from the formulæ (10) given, § 19. Making all the coefficients of u having suffixes vanish, $u_{11} \ldots u_{23} \ldots$ for the present form of u are found; and thence $2\partial^2\sigma/\partial\xi^2 = 4u_{22}u_{33} - u_{23}{}^2 \ldots$ obtained. Next v, expressed in terms of l, m, n as line coordinates, is determined as

$$-\,v \equiv 2\left(l^2\frac{\partial^2\sigma}{\partial\xi^2} + \ldots + 2mn\frac{\partial^2\sigma}{\partial\eta\,\partial\zeta} + \ldots\right),$$

viz., therefrom it is found that

$$v = \Sigma b^2c^2l^6 - 2\,(abc + 16e^3)\,\Sigma am^3n^3 - 24e^2lmn\Sigma\,(bcl^3) - 24\,(abc + 2e^3)\,el^2m^2n^2.$$

Finally, the second differential coefficients of v may be found. And the value of v so found affords an independent verification of the relation (i.), § 21,

$$v \equiv -\,4\sigma,$$

when in σ and v ξ, η, ζ have been replaced by l, m, n.

I may remark, in conclusion, that I first obtained the general form of v by considering the question mentioned in § 35, determining in the resulting form (47) which of the two factors represented v, which ϖ, by examination of the special forms when the line L and the tangents to s at the points $\text{L} = 0$, $s = 0$, were the lines of reference. Subsequently, I observed the method of arriving at the general form of v given §§ 30, 31, independently; which completed the general proof in as simple a manner as could be expected.

38. The triad of tangents $\varpi = 0$ reappear as a part of the complete solution of a

question of considerable interest in connexion with the subject of this Memoir: what, if any, pairs of polar lines of points on L are *conjugate*, in the sense of having a common COTES-point?

The polar line of the point $x'y'z'$ on L, viz.,

$$x\frac{\partial u'}{\partial x'} + y\frac{\partial u'}{\partial y'} + z\frac{\partial u'}{\partial z'} = 0,$$

meets L in the point

$$(x):(y):(z) = n\frac{\partial u'}{\partial y'} - m\frac{\partial u'}{\partial z'} : l\frac{\partial u'}{\partial z'} - n\frac{\partial u'}{\partial x'} : m\frac{\partial u'}{\partial x'} - l\frac{\partial u'}{\partial y'}, \quad . \quad . \quad . \quad (52)$$

and the condition plainly is that the polar line of (x) (y) (z) should pass through the original point $x'y'z'$, *i.e.*, that

$$x'\left(\frac{\partial u}{\partial x}\right) + y'\left(\frac{\partial u}{\partial y}\right) + z'\left(\frac{\partial u}{\partial z}\right) = 0,$$

when for (x) (y) (z) in $(\partial u/\partial x)$ . . . the values just given in terms of $x'y'z'$ (52) are substituted. The result is at once obtained by means of the general theorem, § 24 (and § 27); viz., it is simply

$$u's' = 0,$$

with

$$lx' + my' + nz' = 0.$$

Now, at the three points $u' = 0$, $lx' + my' + nz' = 0$, $[(x)\ (y)\ (z)]$ plainly coincides with $(x'y'z')$; thus each tangent to u at those points is a double conjugate line, which accounts for the factor u' in the result above.

39. The points

$$s' = 0, \qquad lx' + my' + nz' = 0,$$

determine the unique pair of distinct or proper *conjugate polar lines;* viz., these are *the tangents to the poloid at the points in which it is met by the transversal* L; which will be real only when L meets s in real points.

These two conjugate tangents to s and their chord of contact L form, as before remarked, an unique triad of lines, to which the cubic u and its system may be conveniently referred, in questions involving s, v, and certain other concomitants of L and u particularly, as will appear in the sequel. Plainly the equations thus referred will only be symmetric as regards two coordinates, however.

40. Through any point $(x''y''z'')$ in the plane of the cubic u and the transversal L two chords of u can, in general, be drawn having it as their COTES-point, relatively to their intersections with L; viz., the connectors of $(x''y''z'')$ with the points $(x'y'z')$ (. . .) in which its polar conic meets L.

If (xyz) is any other point on one of these chords, then the coordinates of the point in which it meets L, or

$$lx + my + nz = 0,$$

will be proportional to

$$n(x''z - z''x) - m(y''x - x''y) \ldots$$

or

$$\mathrm{L}x'' - \mathrm{L}''x, \qquad \mathrm{L}y'' - \mathrm{L}''y, \qquad \mathrm{L}z'' - \mathrm{L}''z,$$

and the substitution of these coordinates in

$$x^2 \frac{\partial^2 u''}{\partial x''^2} + \ldots + 2yz \frac{\partial^2 u''}{\partial y \, \partial z} + \ldots = 0$$

gives the equation of the two chords in question, in the form

$$(\mathrm{L}x'' - \mathrm{L}''x)^2 \frac{\partial^2 u''}{\partial x''^2} + \ldots + 2(\mathrm{L}y'' - \mathrm{L}''y)(\mathrm{L}z'' - \mathrm{L}''z) \frac{\partial^2 u''}{\partial y'' \, \partial z''} + \ldots = 0,$$

wherein L stands for $lx + my + nz$, L'' for $lx'' + my'' + nz''$.

The equation of that one of the pair which cuts L in the point $(x'y'z')$ being

$$(y'z'' - z'y'')(\mathrm{L}x'' - \mathrm{L}''x) + (z'x'' - x'z'')(\mathrm{L}y'' - \mathrm{L}''y) + (x'y'' - y'x'')(\mathrm{L}z'' - \mathrm{L}''z) = 0,$$

that of the other will be either

$$\begin{aligned} &(y'z'' - z'y'') \frac{\partial^2 u''}{\partial x''^2} (\mathrm{L}x'' - \mathrm{L}''x) \\ &+ \left\{ (x'z'' - z'x'') \frac{\partial^2 u''}{\partial x''^2} + 2(y'z'' - z'y'') \frac{\partial^2 u''}{\partial x'' \, \partial y''} \right\} (\mathrm{L}y'' - \mathrm{L}''y) \\ &+ \left\{ (y'x'' - x'y'') \frac{\partial^2 u''}{\partial x''^2} + 2(y'z'' - z'y'') \frac{\partial^2 u''}{\partial z'' \, \partial x''} \right\} (\mathrm{L}z'' - \mathrm{L}''z) = 0, \quad . \quad . \quad (53) \end{aligned}$$

or one of the two analogous forms obtained by interchanging x'', y'' or x'', z''.

41. The reciprocal of the equation (52) being, to a factor,

$$\left\{ \frac{\partial^2 u''}{\partial y''^2} \frac{\partial^2 u''}{\partial z''^2} - \left(\frac{\partial^2 u''}{\partial y'' \, \partial z''} \right)^2 \right\} l^2 + \ldots + 2 \left(\frac{\partial^2 u''}{\partial z'' \, \partial x''} \frac{\partial^2 u''}{\partial x'' \, \partial y''} - \frac{\partial^2 u''}{\partial x''^2} \frac{\partial^2 u''}{\partial y'' \, \partial z''} \right) mn + \ldots = 0,$$

if $x''y''z''$ should be a point on the poloid s, the two chords having it as their COTES-point will coincide—as would otherwise be inferred from the consideration that any point on s might be regarded as the point of intersection of two coincident tangents—or become a "double chord," and its equation will be either

$$(\mathrm{L}x'' - \mathrm{L}''x) \frac{\partial^2 u''}{\partial x''^2} + (\mathrm{L}y'' - \mathrm{L}''y) \frac{\partial^2 u''}{\partial x'' \, \partial y''} + (\mathrm{L}z'' - \mathrm{L}''z) \frac{\partial^2 u''}{\partial z'' \, \partial x''} = 0, \quad . \quad (54)$$

or one of two analogous forms resulting from the interchange of x'', y'' or x'', z''.

Otherwise, the equation of the double chord through $(x''y''z'')$, a point on s is

$$2\mathrm{L} \frac{\partial u''}{\partial x''} - \mathrm{L}'' \left(x \frac{\partial}{\partial x''} + y \frac{\partial}{\partial y''} + z \frac{\partial}{\partial z''} \right) \frac{\partial u''}{\partial x''} = 0. \quad . \quad . \quad . \quad . \quad . \quad (55)$$

42. But considering a point $(x'y'z')$ on L, the double chord through it is, plainly, that connecting it with the point in which its polar line touches the envelope s, viz., this is the polar with respect to s of the COTES-point on that polar-line (§ 29). Its equation, thus considered, is therefore

$$x\left(\frac{\partial s}{\partial x}\right) + y\left(\frac{\partial s}{\partial y}\right) + z\left(\frac{\partial s}{\partial z}\right) = 0,$$

with the coordinates of the COTES-point referred to substituted for xyz in $(\partial s/\partial x)$ $(\partial s/\partial y)$ $(\partial s/\partial z)$; those coordinates being (36), § 29,

$$\frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial y'} - \frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial z'}, \quad \frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial z'} - \frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial x'}, \quad \frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial x'} - \frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial y'}.$$

The resulting equation to the double chord is, therefore,

$$\left(\frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial y'} - \frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial x'}\right)\frac{\partial s}{\partial x} + \left(\frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial z'} - \frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial x'}\right)\frac{\partial s}{\partial y} + \left(\frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial x'} - \frac{\partial s'}{\partial x'}\frac{\partial u'}{\partial y'}\right)\frac{\partial s}{\partial z} = 0. \quad . \quad (56)$$

43. This being cubic in $x'y'z'$, and of five dimensions in the coefficients of u, its envelope, obtained as the condition that $lx' + my' + nz' = 0$ shall touch the ternary cubic in $x'y'z'$, will be a curve of order 4 in xyz, of order 6 in lmn, and of order* 8 in the coefficients of u.

This envelope, as well as that of the complex of chords of u, connected with the polar line of a point on L by having their COTES-points on it, but not passing through its pole, will be more conveniently considered by means of special forms of s and u, to which every cubic may be reduced.

---

* This would appear to be 20, but another form of the equation of the double-chord, into which the coefficients of u enter only in the *second* degree, may be obtained from that of s (8 bis) given in the Note to § 22. Combining (i) and (iii) of that Note, the coordinates of the point of contact of the polar line of $(x'y'z')$ with the poloid (D, fig. 1) are at once given as proportional to

$$u'_2v'_3 - u'_3v'_2, \quad u'_3v'_1 - u'_1v'_3, \quad u'_1v'_2 - u'_2v'_1,$$

respectively; hence, the line joining this point with $(x'y'z')$ is

$$u'(x\,\mathrm{D}u'_1 + y\,\mathrm{D}u'_2 + z\,\mathrm{D}u'_3) - \mathrm{D}u'(xu'_1 + yu'_2 + zu'_3) = 0.$$

But, if $6a \ldots 6f \ldots$ stand for $\partial^2u/\partial x^2 \ldots \partial^2u/\partial y\,\partial z \ldots$; λ, μ, ν for $\gamma m - \beta n$, $\alpha n - \gamma l$, $\beta l - \alpha m$, respectively,

$$\mathrm{D}u'_1 = \lambda a' + \mu h' + \nu g', \quad \mathrm{D}u'_2 = \lambda h' + \mu b' + \nu f', \quad \mathrm{D}u'_3 = \lambda g' + \mu f' + \nu c',$$

$$u' = x'u'_1 + y'u'_2 + z'u'_3, \quad xu'_1 + yu'_2 + zu'_3 = x'^2a + \ldots + 2y'z'f + \ldots;$$

while, since $lx' + my' + nz' = 0$

$$z'\mu - y'\nu = z'(\alpha n - \gamma l) - y'(\beta l - \alpha m) = -l(\alpha z' + \beta y' + \gamma z') = -l\Delta'; \quad x'\nu - z'\lambda = -m\Delta'; \quad y'\lambda - x'\mu = -n\Delta'.$$

Substituting in (i) the equation of the double-chord (DE, fig. 1) becomes divisible by Δ', and is

$$\left(n\frac{\partial u'}{\partial y'} - m\frac{\partial u'}{\partial z'}\right)\left(x'\frac{\partial^2u}{\partial x^2} + y'\frac{\partial^2u}{\partial x\,\partial y} + z'\frac{\partial^2u}{\partial z\,\partial x}\right) + \left(l\frac{\partial u'}{\partial z'} - n\frac{\partial u'}{\partial x'}\right)\left(x'\frac{\partial^2u}{\partial x\,\partial y} + y'\frac{\partial^2u}{\partial y^2} + z'\frac{\partial^2u}{\partial y\,\partial z}\right)$$

$$+ \left(m\frac{\partial u'}{\partial x'} - l\frac{\partial u'}{\partial y'}\right)\left(x'\frac{\partial^2u}{\partial z\,\partial x} + y'\frac{\partial^2u}{\partial y\,\partial z} + z'\frac{\partial^2u}{\partial z^2}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\text{ bis})$$

—(Note added April, 1888.)

44. The question of the angle (θ) between the polar line of a point ($x'y'z'$) on L and the "double" chord of the pencil through that point is of some interest, as a generalisation of the question of the angle which a Newtonian "diameter" makes with its "ordinates."

The polar line being

$$x\frac{\partial u'}{\partial x'} + y\frac{\partial u'}{\partial y'} + z\frac{\partial u'}{\partial z'} = 0,$$

and the double chord (56)

$$\Sigma\left(\frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial y'} - \frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial z'}\right)\frac{\partial s}{\partial x} = 0,$$

what may be called the "direction coordinates" of the two lines are (the angles of the triangle of reference being A, B, C)

$$\sin C\frac{\partial u'}{\partial y'} - \sin B\frac{\partial u'}{\partial z'}, \quad \sin A\frac{\partial u'}{\partial z'} - \sin C\frac{\partial u'}{\partial x'}, \quad \sin B\frac{\partial u'}{\partial x'} - \sin A\frac{\partial u'}{\partial y'}$$

and, using Δ' as in § 20 ; A . . . H as in § 30,

$$\begin{aligned}
&x'\left\{(\mathrm{A}\sin A+\ldots)\frac{\partial u'}{\partial x'}+(\mathrm{H}\sin A+\ldots)\frac{\partial u'}{\partial y'}+(\mathrm{G}\sin A+\ldots)\frac{\partial u'}{\partial z'}\right\}-\Delta'\left(\mathrm{A}\frac{\partial u'}{\partial x'}+\mathrm{H}\frac{\partial u'}{\partial y'}+\mathrm{G}\frac{\partial u'}{\partial z'}\right),\\
&y'\left\{\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\right\}-\Delta'\left(\mathrm{H}\frac{\partial u'}{\partial x'}+\mathrm{B}\frac{\partial u'}{\partial y'}+\mathrm{F}\frac{\partial u'}{\partial z'}\right),\\
&z'\left\{\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\right\}-\Delta'\left(\mathrm{G}\frac{\partial u'}{\partial x'}+\mathrm{F}\frac{\partial u'}{\partial y'}+\mathrm{C}\frac{\partial u'}{\partial z'}\right).
\end{aligned}$$

The angle between two lines whose direction coordinates are $\lambda\mu\nu$, $\lambda'\mu'\nu'$ is given, generally, by

$$\tan\theta = \Sigma(\mu\nu' - \mu'\nu)\sin B\sin C/3\Sigma\lambda\lambda'\sin A\cos A, \quad . \quad . \quad . \quad (57)$$

and in the particular case now under consideration

$$\begin{aligned}
\tan\theta = \sin A\sin B\sin C&\left[u'\Sigma(\mathrm{A}\sin A+\ldots)\frac{\partial u'}{\partial x'}-\Delta'\left\{\mathrm{A}\left(\frac{\partial u'}{\partial x'}\right)^2+\ldots+2\mathrm{F}\frac{\partial u'}{\partial y'}\frac{\partial u'}{\partial z'}+\ldots\right\}\right]\\
&\div\left[\left\{(\mathrm{A}\sin A+\ldots)\frac{\partial u'}{\partial x'}+(\mathrm{H}\sin A+\ldots)\frac{\partial u'}{\partial y'}\right.\right.\\
&\qquad\left.+(\mathrm{G}\sin A+\ldots)\frac{\partial u'}{\partial z'}\right\}\Sigma x'\sin A\cos A\left(\sin C\frac{\partial u'}{\partial y'}-\sin B\frac{\partial u'}{\partial z'}\right)\\
&\qquad\left.-\Delta'\Sigma\left(\mathrm{A}\frac{\partial u'}{\partial x'}+\mathrm{H}\frac{\partial u'}{\partial y'}+\mathrm{G}\frac{\partial u'}{\partial z'}\right)\sin A\cos A\left(\sin C\frac{\partial u'}{\partial y'}-\sin B\frac{\partial u'}{\partial z'}\right)\right]. \quad . \quad (58)
\end{aligned}$$

45. If $x''y''z''$ is the point of contact with its envelope, the poloid s, of the polar line of a point $x'y'z'$ on L or

$$lx + my + nz = 0,$$

from the forms of the equation of that polar line it is evident that

$$\frac{\partial s''}{\partial x''} : \frac{\partial s''}{\partial y''} : \frac{\partial s''}{\partial z''} = \frac{\partial u'}{\partial x'} : \frac{\partial u'}{\partial y'} : \frac{\partial u'}{\partial z'},$$

whence at once x'', y'', z'' are given in terms of $x'y'z'$ by

$$\begin{array}{ccccc} x'' & : & y'' & : & z' \end{array}$$

$$= \mathrm{A}\frac{\partial u'}{\partial x'} + \mathrm{H}\frac{\partial u'}{\partial y'} + \mathrm{G}\frac{\partial u'}{\partial z'} : \mathrm{H}\frac{\partial u'}{\partial x'} + \mathrm{B}\frac{\partial u'}{\partial y'} + \mathrm{F}\frac{\partial u'}{\partial z'} : \mathrm{G}\frac{\partial u'}{\partial x'} + \mathrm{F}\frac{\partial u'}{\partial y'} + \mathrm{C}\frac{\partial u'}{\partial z'}, \quad . \quad (59)$$

A . . . 2F . . . being the coefficients of the reciprocal of s.

Conversely, if x', y', z' are to be expressed in terms of x'', y'', z'', they are the coordinates of the point common to the two conics

$$\frac{\partial s}{\partial z}\frac{\partial u}{\partial y} - \frac{\partial s}{\partial y}\frac{\partial u}{\partial z} = \frac{\partial s}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial s}{\partial z}\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial s}{\partial x}\frac{\partial u}{\partial y}$$

and the line

$$lx + my + nz = 0.$$

46. But the point $x'y'z'$ is more readily determined by the double line of the pencil of chords through that point, having their Cotes-points on the tangent to the poloid at x, y, z, which has been shown to be (54)

$$(\mathrm{L}x'' - \mathrm{L}''x)\frac{\partial^2 u''}{\partial x''^2} + (\mathrm{L}y'' - \mathrm{L}''y)\frac{\partial^2 u''}{\partial x''\,\partial y''} + (\mathrm{L}z'' - \mathrm{L}''z)\frac{\partial^2 u''}{\partial x''\,\partial z''} = 0.$$

At the intersection of this and L

$$x\frac{\partial^2 u''}{\partial x''^2} + y\frac{\partial^2 u''}{\partial x''\,\partial y''} + z\frac{\partial^2 u''}{\partial x''\,\partial z''} = 0,$$

giving

$$\begin{array}{ccccc} x' & : & y' & : & z' \end{array}$$

$$= n\frac{\partial^2 u''}{\partial x''\,\partial y''} - m\frac{\partial^2 u''}{\partial x''\,\partial z''} : l\frac{\partial^2 u''}{\partial x''\,\partial z''} - n\frac{\partial^2 u''}{\partial x''^2} : m\frac{\partial^2 u''}{\partial^2 x''^2} - l\frac{\partial^2 u''}{\partial x''\,\partial y''}, \quad . \quad . \quad (60)$$

or either of two other analogous sets of values.

47. Thus, when L is taken as $z = 0$, and

$$u \equiv ax^3 + by^3 + cz^3 + 3c_1z^2x + 3c_2z^2y + 6exyz,$$
$$s \equiv -e^2z^2 + abxy,$$

of the coefficients A . . . 2H of the tangential equations of the poloid s, all vanish except

$$4\mathrm{C} = -a^2b^2, \qquad 4\mathrm{H} = 2abe^2;$$

giving for the point of contact of the polar line of $(x'y'o)$

$$x'' : y'' : z'' = bey'^2 : aex'^2 : -abx'y' ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and conversely

$$x' : y' = -ez'' : ax'' = by'' : -ez''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

IV. Special Forms of Cubic and Poloid.

48. I now treat some of the questions, hitherto considered without reference to any particular form of u, in more detail, by means of special forms in which the line L is used as one of the lines of reference--say

$$\mathrm{L} \equiv nz = 0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

and for the two other lines of reference I take 1st, the tangents to the poloid s from the pole of L, as

$$x = 0, \qquad y = 0.$$

The conic s is now (10) § 19—n^2 being dropped—reduced to

$$(a_3b_3 - e^2)z^2 + (ab - a_2b_1)xy = 0,$$

with the conditions

$$ab_1 - a_2{}^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

$$ba_2 - b_1{}^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.})$$

$$ba_3 + a_2b_3 - 2b_1e = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iii.})$$

$$ab_3 + a_3b_1 - 2a_2e = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iv.})$$

From the first and second of these,

$$a_2(ab - a_2b_1) = 0,$$

whence

$$a_2 = 0,$$

since $ab - a_2b_1 = 0$ is excluded by the form of s.

Hence, from the second

$$b_1 = 0,$$

and from the third and fourth

$$a_3 = 0, \qquad b_3 = 0,$$

since $b = 0$ or $a = 0$ is excluded by the form of s, now reduced to

$$s/n^2 \equiv - e^2z^2 + abxy, \qquad (64)$$

and u is reduced to the terms

$$ax^3 + by^3 + cz^3 + 3c_1z^2x + 3c_2z^2y + 6exyz. \qquad (65)$$

49. The line $z = 0$ meets u in the points

$$x : y = b^{\frac{1}{3}} : - a^{\frac{1}{3}}, \quad x : y = \vartheta b^{\frac{1}{3}} : - \vartheta' a^{\frac{1}{3}}, \quad x : y = \vartheta' b^{\frac{1}{3}} : - \vartheta a^{\frac{1}{3}}$$

(ϑ, ϑ' being the imaginary cube roots of $- 1$, so that

$$\vartheta + \vartheta' = 1, \quad \vartheta\vartheta' = 1, \quad \vartheta^2 = - \vartheta', \quad \vartheta'^2 = - \vartheta, \quad \vartheta^2 + \vartheta'^2 = - 1),$$

and the tangents to u at these points are

$$\left.\begin{aligned} a^{\frac{2}{3}}b^{\frac{1}{3}}x + a^{\frac{1}{3}}b^{\frac{2}{3}}y - 2ez, \\ \vartheta a^{\frac{2}{3}}b^{\frac{1}{3}}x + \vartheta' a^{\frac{1}{3}}b^{\frac{2}{3}}y + 2ez, \\ \vartheta' a^{\frac{2}{3}}b^{\frac{1}{3}}x + \vartheta a^{\frac{1}{3}}b^{\frac{2}{3}}y + 2ez\,; \end{aligned}\right\} \qquad (66)$$

the common equation to the three being, by actual multiplication,

$$a^2bx^3 + ab^2y^3 - 8e^3z^3 + 6abexyz = 0, \qquad (67)$$

or, restoring lapsed factors,

$$\varpi \equiv a^2b^2n^6u - \{3a^2b^2c_1n^4x + 3a^2b^2c_2n^4y + (a^2b^2cn^4 + 8abe^3n^4)z\}n^2z^2 = 0. \qquad (68)$$

Further, the same equation may be thrown into the form

$$\varpi \equiv a^2b^2n^6 (ax^3 + by^3 - 2exyz) + 8aben^3 (- e^2n^2z^2 + abn^2xy)\, nz = 0. \qquad (69)$$

50. The tangential equation of s is now simply

$$\mathrm{C}\zeta^2 + 2\mathrm{H}\xi\eta = 0,$$

where

$$4\mathrm{C} = - a^2b^2n^4,$$
$$4\mathrm{H} = 2abe^2n^4.$$

The general value (37) § 30, of v, the Cotesian of L, now reduces to

$$\begin{aligned} v \equiv & - (ax^3 + by^3 + cz^3 + 3c_1z^2x + 3c_2z^2y + 6exyz)\,(-a^2b^2n^6) \\ & - n^2z^2\{-a^2b^2n^4\,(c_1x + c_2y + cz) + 2abe^2n^4\,(ez)\} \\ & + 2nz\{-a^2b^2n^5\,(cz^2 + 2c_1zx + 2c_2yz)\} \\ & + 2\,(e^2n^2z^2 + abn^2xy)\,2\,(-abn^2)\,ezn^2, \end{aligned}$$

i.e.,

$$v \equiv a^2b^2n^6\,(ax^3 + by^3 - 2exyz), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (70)$$

a cubic having a node at the pole of L (now $z = 0$) with respect to s, the tangents to s being also the nodal tangents of the Cotesian v.

51. The comparison of the equations (69) (70) of ϖ and v verifies the relation, (48) § 37,

$$\varpi \equiv v + 8\mathrm{PL}s,$$

since, in the present form (65) of u, the Cayleyan P, for the values of the line coordinates $l = 0$, $m = 0$, is simply

$$\mathrm{P} = aben^3. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (71)$$

52. Eliminating y and x successively between

$$v \equiv ax^3 + by^3 - 2exyz = 0,$$
$$s \equiv -e^2z^2 + abxy = 0,$$

there result

$$(a^2bx^3 - e^3z^3)^2 = 0,$$
$$(ab^2y^3 - e^3z^3)^2 = 0,$$

showing that v has triple contact with s; viz., when the loop of v is real it touches s at the point

$$x : y : z = (a^2b)^{-\frac{1}{3}} : (ab^2)^{-\frac{1}{3}} : e^{-1}\,;$$

or at the point determined by the real lines, or any pair of them,

$$(a^2b)^{\frac{1}{3}}x = (ab^2)^{\frac{1}{3}}y = ez.$$

Now, making $z = 0$ in u and v, the points in which L meets these cubics are determined by

$$ax^3 + by^3 = 0,$$

the real one being

$$a^{\frac{1}{3}}x + b^{\frac{1}{3}}y = 0,$$
$$z = 0,$$

so that *the connectors of the pole of* L (with respect to s) *with this point and the point of contact of v with s are harmonic conjugates relatively to the tangents to s from that pole.*

53. The polar line of P $(x'y'o)$, a point on L, meets the Cotesian v in three points (say, D_0 its COTES-point) and two other points (say, D_1, D_2), while it touches the poloid s in a point D. Now D_0 is also the COTES-point on this polar line relatively to the point D and the cubic v: to prove this—

The coordinates of D are, (59), § 45,

$$x : y : z = bey'^2 : aex'^2 : -abx'y', \quad \ldots \ldots \quad (72)$$

and

$$\frac{\partial v}{\partial x} : \frac{\partial v}{\partial y} : \frac{\partial v}{\partial z} = 3ax^2 - 2eyz : 3by^2 - 2ezx : -2exy \quad \ldots \quad (73)$$

becomes, for those values of the coordinates of D,

$$\frac{\partial v}{\partial x} \quad : \quad \frac{\partial v}{\partial y} \quad : \quad \frac{\partial v}{\partial z}$$

$$= 3by'^4 + 2ax'^3y' : 3ax'^4 + 2bx'y'^3 : -2ex'^2y'^2; \quad \ldots \quad (74)$$

viz., the v-polar line of D is

$$3(ax'^3 + by'^3)(y'x + x'y) - x'y'(ax'^2x + by'^2y + 2ex'y'z) = 0; \quad \ldots \quad (75)$$

but

$$ax'^2x + by'^2y + 2ex'y'z = 0 \quad \ldots \ldots \ldots \quad (76)$$

is the u-polar line of $(x'y'o)$, and

$$y'x + x'y = 0, \quad \ldots \ldots \ldots \ldots \quad (77)$$

the s-polar of the same point, cuts it in its COTES-point, D_0 (35), § 28. Thus the three lines (75) (76) (77) meet in one point (D_0), and this is the u-COTES-point relatively to P, and the v-COTES-point to D of (76); whence

$$\frac{3}{DD_0} = \frac{1}{DD_0} + \frac{1}{DD_1} + \frac{1}{DD_2}$$

or

$$\frac{2}{DD_0} = \frac{1}{DD_1} + \frac{1}{DD_2},$$

as verified very exactly in figs. 2, 3, *i.e., the* COTES-*point on the polar line of a point in* L *is harmonic-conjugate to its point of contact with the poloid of* L *and u relatively* *to the two other points in which it meets the* Cotesian v; and is thus discriminated from them, when all three are real. If one only is real, that of course is the COTES-point.

54. The general form given above (49), § 35, for the satellite chord of the line L, may be verified from the form it takes for the present one of u (65), the reciprocal of which—as far as the terms giving those which do not vanish in its second differential coefficients for $\xi = 0$, $\eta = 0$, $\zeta = n$—is

$$v = a^2b^2\zeta^6 + 6ab^2c_1\zeta^4\xi^2 + 6a^2bc_2\zeta^4\eta^2 - 24abe^2\zeta^4\xi\eta\,; \quad . \quad . \quad . \quad (78)$$

viz., for those values of ξ, η, ζ,

$$\tfrac{1}{6}\frac{\partial^2 v}{\partial\xi^2} = 2ab^2c_1n^4,$$

$$\tfrac{1}{6}\frac{\partial^2 v}{\partial\eta^2} = 2a^2bc_2n^4,$$

$$\tfrac{1}{6}\frac{\partial^2 v}{\partial\zeta^2} = 5a^2b^2n^4,$$

$$\tfrac{1}{6}\frac{\partial^2 v}{\partial\xi\,\partial\eta} = -\,4abe^2n^4\,;$$

while the reciprocal of s ($\equiv -\,e^2z^2 + abxy$) being now

$$4\sigma \equiv -\,a^2b^2n^4\zeta^2 + 4abe^2n^4\xi\eta,$$

$$8\frac{\partial^2\sigma}{\partial\zeta^2} = -\,4a^2b^2n^4,$$

$$8\frac{\partial^2\sigma}{\partial\xi\,\partial\eta} = 8abe^2n^4.$$

Also

$$\frac{\partial^2 u}{\partial x^2} = 6ax, \qquad \frac{\partial^2 u}{\partial y^2} = 6by, \qquad \frac{\partial^2 u}{\partial z^2} = 6\,(c_1x + c_2y + cz),$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 6ez, \qquad \frac{\partial^2 u}{\partial z\,\partial x} = 6\,(ey + c_1z), \qquad \frac{\partial^2 u}{\partial z\,\partial y} = 6\,(ex + c_2z).$$

With these values of the second differential coefficients, which do not vanish, for the present forms (64) (65) the satellite chord of L, or $nz = 0$, is n^4 multiplied by

$$\begin{aligned}(2ab^2c_1 + *)\,ax + (2a^2bc_2 + *)\,by + (5a^2b^2 - 4a^2b^2)\,(c_1x + c_2y + cz)&\\ + 2(-\,4abe^2 + 8abe^2)\,ez + (* + *)\,(ey + c_1z) + (* + *)\,(ex + c_2z)&,\end{aligned}$$

or, identically,

$$3a^2b^2c_1x + 3a^2b^2c_2y + (a^2b^2c + 8abe^3)\,z,$$

which agrees with the form in ϖ (68).

55. What has preceded is, substantially, a verification of the relation (39) § 31; but, to make it more clear, observing that the only one of the coefficients A . . . F . . . of the reciprocal of s which does not disappear from the sinister of (39) is C, in the term

$$96\mathrm{C}n\frac{\partial u}{\partial z} = -24a^2b^2n^5(3cz^2 + 6c_1zx + 6c_2yz + 6exy),$$

and adding to this

$$144\mathrm{P}s = 144aben^3(-e^2n^2z^2 + abn^2xy),$$

that sinister becomes

$$-72nz\{2a^2b^2n^4c_1x + 2a^2b^2n^4c_2y + (a^2b^2n^4c + 2abe^2n^4e)z\}.$$

Again, the dexter is (L being now nz)

$$-nz\{(12ab^2c_1n^4)6ax + (12a^2bc_2n^4)6by + (30a^2b^2n^4 - 18a^2b^2n^4)6(c_1x + c_2y + cz) \\ + 2(-24abe^2n^4 + 36abe^2n^4)6ez\},$$

or, identically, also

$$-nz\{144a^2b^2n^4(c_1x + c_2y) + (72a^2b^2n^4c + 144abe^2n^4e)z\}.$$

Lastly,

$$-4u_{12}n^2\frac{\partial^2 u}{\partial x\,\partial y}/6,$$

the only term in the sinister of (38),

$$= -4aben^4z = (71)\ §\ 51,\ -4\mathrm{PL}.$$

56. It has been shown that v and u meet the transversal L in the same three points (§ 52),

$$ax^3 + by^3 = 0, \qquad z = 0,$$

of which one only is real when the line L meets its poloid s in real points, the condition of the lines of reference $x = 0$, $y = 0$, at present used, being real. That this would be the case appeared from the fact that the discriminants of the cubic (12) and quadratic (13) § 20, whose roots are proportional to the intercepts of L between a certain point on it and its intersections with u and s respectively, are of opposite signs.

57. The tangents to v at the points in which it is met by L are plainly obtained from those to u at the same points, by changing e into $-e/3$; viz., they are (66)

$$3a^{\frac{2}{3}}b^{\frac{1}{3}}x + 3a^{\frac{1}{3}}b^{\frac{2}{3}}y + 2ez = 0\ .\ .\ .$$

eliminating z between this equation and v, the result is

$$(a^{\frac{1}{3}}x + b^{\frac{1}{3}}y)^3 = 0,$$

viz., the points in which v is met by L are points of inflexion on v, as was manifest from the form of its equation. When the node of v is real, only one of these points of inflexion is real; but if v is acnodal, all three will be points of real inflexion. Thus much is known of v generally. In Plates 7, 8, it is figured for two cases when L is the line at infinity, as will be further described.

58. The coordinates of the COTES-point on the polar line of $(x'y'0)$

$$ax'^2 x + by'^2 y + 2ex'y'z = 0$$

may be found as those of its point of intersection with

$$y'z + z'y = 0,$$

the s-polar of $(x'y'0)$; viz., they are

$$x : y : z = - 2ex'^2y' : 2ex'y'^2 : ax'^3 - by'^3 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

or, what is the same thing (36), p. 165,

$$x : y : z = \frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial y'} - \frac{\partial s'}{\partial y'}\frac{\partial u}{\partial z'} : \ldots : \ldots$$

The double chord (56) §42, through $(x'y'0)$ is determined as the polar with respect to s of that COTES-point (79) on the polar line of $(x'y'0)$. Its equation is, therefore,

$$abx'y'(- y'x + x'y) + e(ax'^3 - by'^3)z = 0.$$

Arranging this as a binary cubic in $x'y'$,

$$aezx'^3 + abyx'^2y' - abxx'y'^2 - bezy'^3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

the discriminant will be the envelope of the double chord as the point $(x'y'0)$ varies on the transversal L, viz.,

$$3w \equiv 4ab\,(ax^2 + 3eyz)\,(by^2 + 3exz) - (abxy - 9e^2z^2)^2 = 0, \quad . \quad . \quad . \quad (81)$$

or, developed,

$$w \equiv a^2b^2x^2y^2 + 4a^2bex^3z + 4ab^2ey^3z + 18abe^2xyz^2 - 27e^4z^4 = 0.$$

59. If the first derived of the cubic (80), *i.e.*,

$$\left.\begin{aligned} 3ex'^2z + 2bx'y'x - by'^2x = 0, \\ 3ey'^2z + 2ax'y'x - ax'^2y = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

are combined, the coordinates of the point of contact of the double chord with its envelope will be obtained in terms of those $(x'y')$ of the point on L, through which it is drawn, viz., eliminating z between the equations (82), the point of contact is determined by the line

$$(2ax'^3 + by'^3)\, y'x - (ax'^3 + 2by'^3)\, x'y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

The intersection of the double chord with L lies on the line

$$y'x - x'y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

and its two intersections with the poloid on the lines

$$by'^4x - ax'^4y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

$$ax'^2x - by'^2y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

the latter being its Cotes-point, viz., the point of contact with s of the polar line of $(x', y', 0)$—(61), § 47.

Now these four lines through $x = 0$, $y = 0$, form a harmonic pencil, since, writing (84) (86)

$$y'x - x'y = p,$$
$$ax'^2x - by'^2y = q,$$

[giving

$$(ax'^3 - by'^3)\, x = - by'^2p + x'q,$$
$$(ax'^3 - by'^3)\, y = - ax'^2p + y'q]$$

then (85)

$$by'^4x - ax'^4y = (a^2x^6 - b^2y^6)\, p + (by'^3 - ax'^3)\, x'y'q,$$

and (83)

$$\begin{aligned}(2ax'^3 + by'^3)\, y'x - (2by'^3 + ax'^3)\, x'y = {} & \{(- 2ax'^3 - by'^3)\, by'^3 + (2by'^3 + ax'^3)\, ax'^3\}\, p \\ & + \{(2ax'^3 + by'^3)\, x'y - (2by'^3 + ax'^3)\, x'y'\}\, q \\ = {} & (a^2x'^6 - b^2y'^6)\, p + (ax'^3 - by'^3)\, x'y'q\,;\end{aligned}$$

viz., the later pair are harmonic-conjugate with respect to the former pair p, q. Thus it appears that *the* Cotes-*point on a double chord, and its intersection with the line* L, *are harmonic-conjugate points with respect to its second (or "empty") intersection with the poloid s and its contact with its envelope w,* as stated, § 7. This is shown in Plates 7, 8, in the case in which L is the line at infinity, by D being the mid-point of CC_1.

60. Returning to the equation (81) of the quartic w, if y and x be alternately eliminated from either

$$ax^2 + 3eyz = 0, \qquad \text{or} \qquad by^2 + 3ezx = 0,$$

and

$$abxy - 9e^2z^2 = 0,$$

there result

$$a^2bx^3 + 27e^3z^3 = 0,$$
$$ab^2y^3 + 27e^3z^3 = 0,$$

showing that w is a tricuspidal quartic, two of the cusps being imaginary when the line L meets the cubic u in one real point only, the real cusp being at the point

$$x : y : z = 3eb^{\frac{1}{3}} : 3ea^{\frac{1}{3}} : -a^{\frac{2}{3}}b^{\frac{2}{3}}$$

on the line

$$a^{\frac{1}{3}}x - b^{\frac{1}{3}}y = 0,$$

which is *the cuspidal tangent,* and *is harmonic-conjugate to the connector of the s-pole of* L *and its real intersection with u, relatively to the tangents to s from that pole.*

The curve w is shown in the figure, Plate 7, in the case when L is at infinity and the cubic has a single real asymptote.

61. Through any point $x''y''z''$ on the polar line of a point $x'y'z'$ in L, two chords of u pass which have the first $(x''y''z'')$ as their Cotes-point; viz., the chord which forms one of the pencil through $x'y'z'$ and the other (53), § 40 the line,

$$\begin{aligned}&\frac{\partial^2 u}{\partial x''^2}(y'z'' - z'y'')\,(\mathrm{L}x'' - \mathrm{L}''x)\\ &\quad + \left\{-\frac{\partial^2 u}{\partial x''^2}(z'x'' - x'z'') + 2\frac{\partial^2 u}{\partial x''\,\partial y''}(y'z'' - z'y'')\right\}(\mathrm{L}y'' - \mathrm{L}''y)\\ &\quad + \left\{-\frac{\partial^2 u}{\partial x''^2}(x'y'' - y'x'') + 2\frac{\partial^2 u}{\partial x''\partial z''}(y'z'' - z'y'')\right\}(\mathrm{L}z'' - \mathrm{L}''z) = 0,\end{aligned}$$

$x''y''z''$ being subject to the relation

$$x''\frac{\partial u}{\partial x'} + y''\frac{\partial u}{\partial y'} + z''\frac{\partial u}{\partial z'} = 0\,;$$

and the envelope of this line, or "alien" chord, may be found for any assigned form of u.

In the case when L is $z = 0$,

$$s \equiv -e^2z^2 + abxy,$$

$$u \equiv ax^3 + by^3 + cz^3 + 3c_1z^2x + 3c_2z^2y + 6exyz,$$

$$\tfrac{1}{6}\frac{\partial^2 u}{\partial x''^2} = ax'', \quad \tfrac{1}{6}\frac{\partial^2 u}{\partial x''\partial y''} = ez'', \quad \tfrac{1}{6}\frac{\partial^2 u}{\partial x''\partial z''} = ey'',$$

and

$$\tfrac{1}{3}\frac{\partial u}{\partial x'} = ax'^2, \quad \tfrac{1}{3}\frac{\partial u}{\partial y'} = by'^2, \quad \tfrac{1}{3}\frac{\partial u}{\partial z'} = 2ex'y',$$

since

$$z' = 0.$$

The equation of the "alien" chord is then simply

$$ax''z''y'(zx'' - xz'') + (ax''z''x' + 2ez''^2y')(zy'' - yz'') = 0$$

with (76)

$$ax'^2x'' + by'^2y'' + 2ex'y'z'' = 0,$$

in virtue of which the equation of the "alien" chord reduces to

$$ax'x''(xz'' - zx'') - by'y''(yz'' - zy'') = 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

the envelope of which is

$$ab\,(y'x + x'y)^2 + 8e\,(ax'^2x + by'^2y + 2ex'y'z)\,z = 0, \quad . \quad . \quad . \quad (88)$$

a conic touching the line L at the point which is harmonic-conjugate to $(x'y'0)$ with respect to the intersections of L with the poloid s; and touching the polar line of $(x'y'0)$ at the point in which it is met by

$$y'x + x'y = 0,$$

viz., the point

$$x : y : z = -\,2ex'^2y' : 2ex'y'^2 : ax'^3 - by'^3,$$

i.e., at the COTES-point on that polar line (79), § 48.

62. The conic envelope just found may be called the "satellite-conic" to the polar line of $x'y'z'$. Of course, this conic touches also the double chord of the pencil through $x'y'0$, since it is at once an "alien" and a proper chord of the polar line of $(x'y'0)$.

Arranging the equation (88) to the satellite conic as a quadratic in $x'y'$ it is

$$a\,(by^2 + 8ezx)\,x'^2 + 2\,(abxy + 8e^2z^2)\,x'y' + b\,(ax^2 + 8eyz)\,y'^2 = 0,$$

giving at once, for the envelope of the system of conics satellite to the complex of polar lines of the points on L, the quartic

$$ab\,(ax^2 + 8eyz)\,(by^2 + 8ezx) - (abxy + 8e^2z^2)^2 = 0,$$

which developed is

$$8ez\,(a^2bx^3 + ab^2y^3 - 8e^3z^3 + 6abexyz) = 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

viz., the envelope of the satellite conics is the system of four lines L and the tangents to u at the points $L = 0$, $u = 0$ (67), § 49.

The explanation of this is that at the point of intersection of the polar line (76) with one of the tangents (66), that tangent is itself the "alien" chord for that point: for (76) (66)

$$ax'^2x'' + by'^2y'' + 2ex'y'z'' = 0,$$
$$a^{\frac{2}{3}}b^{\frac{1}{3}}x'' + a^{\frac{1}{3}}b^{\frac{2}{3}}y'' - 2ez'' = 0,$$

give

$$x'' : y'' : z'' = -2b^{\frac{2}{3}}ey' : 2a^{\frac{2}{3}}ex' : a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{1}{3}}x' - b^{\frac{1}{3}}y'),$$

and the substitution of these values in (87) gives for the "alien" chord, after dividing out the extraneous factors $2abex'y'(a^{\frac{1}{3}}x' - b^{\frac{1}{3}}y')$, simply

$$a^{\frac{2}{3}}b^{\frac{1}{3}}x + a^{\frac{1}{3}}b^{\frac{2}{3}}y - 2ez = 0\ ;$$

viz., the tangent to u at the point $x : y :\ z = b^{\frac{1}{3}} : a^{\frac{1}{3}} : 0$.

Thus the system of satellite conics is inscribed in the quadrilateral formed by the line L and the three tangents to u at the points in which L meets it. A satellite parabola is shown in Plate 8, when L is the line at infinity.

63. The tangents to s from the pole of the transversal L being real only when L meets the cubic u in a single real point, it is desirable to use as lines of reference another pair in connexion with L, which shall be real in all cases for purposes in which their reality is essential.

Consider the connector of the pole (C, fig. 1) of L ($z = 0$) with a real point (A) in which L meets u, taking it, say, as $y = 0$; and let the s-polar of the point $z = 0$, $y = 0$, be taken as the third line (BC) of reference, $x = 0$. The equation of s is then reduced to the three terms, remembering that still $l = 0$, $m = 0$,

$$(ab_1 - a_2{}^2)\,x^2 + (ba_2 - b_1{}^2)\,y^2 + (a_3b_3 - e^2)\,z^2 = 0, \quad . \quad . \quad . \quad . \quad (90)$$

since the triangle $x = 0$, $y = 0$, $z = 0$ is self-conjugate with respect to that conic; and as conditions for the terms yz, zx, xy disappearing from its equation

$$\left.\begin{aligned} ba_3 + a_2b_3 &= 2b_1e, \\ ab_3 + a_3b_1 &= 2a_2e, \\ ab - a_2b_1 &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

But since the equation of the three tangents to u at the points

$$u = 0, \qquad z = 0$$

is now of the form

$$ku - k'z^2 = 0,$$

and one of them passes through the point

$$y = 0, \qquad z = 0,$$

the term x^3 must disappear from u, or

$$a = 0,$$

which (91) necessitates either

$$a_2 \text{ or } b_1 = 0\,;$$

but the supposition $a_2 = 0$ would imply a second of the tangents to u at the points $u = 0$, $z = 0$ passing through the point $y = 0$, $z = 0$, so that

$$b_1 = 0,$$

and therefore, by the second of the three conditions (91)

$$e = 0\,;$$

whence, by the first,

$$b : a_2 = - b_3 : a_3. \quad \text{. . . . .} \quad (92)$$

64. Thus the equation to the cubic is reduced to the terms

$$by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y = 0, \quad \text{. . .} \quad (93)$$

with the relation (92); and (90) the poloid to

$$\left.\begin{aligned} s/n^2 \equiv - a_2{}^2x^2 + ba_2y^2 + a_3b_3z^2 &= 0, \\ \text{or} \qquad ba_2x^2 - b^2y^2 + b_3{}^2z &= 0, \end{aligned}\right\} \quad \text{. . . . .} \quad (94)$$

the reciprocal of which is

$$\left.\begin{aligned} &A\xi^2 + B\eta^2 + C\zeta^2, \\ \text{where} \qquad & \\ &A = ba_2a_3b_3n^4 \quad = - a_3{}^2b_3{}^2n^4, \\ &B = - a_2{}^2a_3b_3n^4 = ba_2a_3{}^2n^4, \\ &C = - ba_2{}^3n^4, \\ \text{or} \qquad & \\ &A : B : C = a_3b_3 : a_3{}^2 : - a_2{}^2. \end{aligned}\right\} \quad \text{. . . . .} \quad (95)$$

65. For the above forms (93) (94) of u and s, the values of their first differential coefficients, with the coordinates $(x'y'0)$ of a point on L substituted for xyz, are

$$\left.\begin{aligned} \frac{\partial s}{\partial x'} : \frac{\partial s}{\partial y'} : \frac{\partial s}{\partial z'} &= - a_2x' : by' : 0, \\ \frac{\partial u}{\partial x'} : \frac{\partial u}{\partial y'} : \frac{\partial u}{\partial z'} &= 2a_2x'y' : by'^2 + a_3x'^2 : a_3x'^2 + b_3y'^2\,; \end{aligned}\right\} \quad \text{. . .} \quad (96)$$

giving (36) for the COTES-point on the u-polar line of that point

$$x : y : z = ba_3x'^2y' + bb_3y'^3 : a_2a_3x'^3 + a_2b_3y'^2x' : - a_2^2x'^3 - 3ba_2y'^2x' \quad . . \quad (97)$$

The values give

$$v \equiv by^3 + 3a_2x^2y - a_3x^2z - b_3x^2z = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (98)$$

in virtue of the relation (92)

$$ba_3 + a_2b_3 = 0 \, ;$$

viz., the above is the equation of the Cotesian for the form (93) of u now used.

66. The Cotesian v for the forms of u and s at present employed (93, 94) has been found quite independently of any application of the general equation (37) given above; but it may be of interest to test the general formula by this result.

The only terms in the general expression which do not disappear in virtue of

$$l = 0, \qquad m = 0, \qquad \mathrm{F} = 0, \qquad \mathrm{G} = 0, \qquad \mathrm{H} = 0,$$

are four times

$$- \mathrm{C}n^2u - \frac{\mathrm{L}^2}{6}\left(\mathrm{A}\frac{\partial^2 u}{\partial x^2} + \mathrm{B}\frac{\partial^2 u}{\partial y^2} + \mathrm{C}\frac{\partial^2 u}{\partial z^2}\right) + \frac{2\mathrm{L}}{3}\mathrm{C}n\frac{\partial u}{\partial z}$$
$$+ \frac{s}{6}\left(u_{22}\frac{\partial^2 u}{\partial x^2} + u_{11}\frac{\partial^2 u}{\partial y^2} - u_{12}\frac{\partial^2 u}{\partial x\, \partial y}\right) n^2,$$

which, taking the values of u, s, A, B, C (93, 94, 95), for the present case, is identically equal to four times

$$\begin{aligned} & ba_2^3n^6\,(by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y) \\ & \quad + n^2z^2\,\{- ba_2a_3b_3n^4\,(a_2y + a_3z) + a_2^2a_3b_3n^4\,(by + b_3z) + ba_2^3n^4\,(c_1x + c_2y + cz)\} \\ & \quad + (- a_2^2x^2 + ba_2y^2 + a_3b_3z^2)\, n^4\,\{ba_2n^2\,(a_2y + a_3z) + (- a_2^2n^2)\,(by + b_3z)\} \\ & \equiv ba_2^3n^6\,(by^3 + 3a_2x^2y + a_2b_3x^2z/b + ba_3y^2z/a_2). \end{aligned}$$

Rejecting the factor $ba_2^3n^6$, and applying the relation (92),

$$a_2b_3/b = - a_3, \qquad \text{or} \qquad ba_3/a_2 = - b_3,$$

this result is, as before (98); or, with the rejected factors restored,

$$v \equiv (by^3 + 3a_2x^2y - a_3x^2z - b_3y^2z) \times 4ba_2^3n^6, \quad . \; . \; . \quad (98 \text{ bis})$$

a cubic having a node at the point

$$x = 0, \qquad y = 0,$$

viz., the s-pole of L or $z = 0$; the nodal tangents being

or (92)
$$\left.\begin{aligned} a_3x^2 + b_3y^2 = 0, \\ a_2x^2 - by^2 = 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (99)$$

The node, therefore, will be *real* or *unreal*, as a_2, b are of the same, or unlike signs.

67. Since, now (94)
$$s \equiv - a_2{}^2x^2 + ba_2y^2 + a_3b_3z^2,$$

it appears that the nodal tangents (99) of v are the tangents to s at the points $s = 0$, L, or $z = 0$, and will therefore only be real when L cuts its poloid in real points, as shown before (§ 50).

The transversal L meets the Cotesian v, as well as the cubic u, at the points

$$z = 0, \qquad y = 0, \qquad z = 0, \qquad by^2 + 3a_2x^2 = 0, \quad . \; . \; . \quad (100)$$

which last two will be real only when v is *acnodal*, or when L does not meet s in real points.

68. The tangents to

$$u \equiv by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y$$

at the points $u = 0$, L, or $z = 0$, are (48)

$$\varpi \equiv v + 8\mathrm{PL}s$$

if, as just above,
$$v/4 \equiv - (\mathrm{A}l^2 + \ldots) u + \ldots;$$
viz., in this case, then,

$$\begin{aligned} \varpi/4 \equiv\; & ba_2{}^3n^6 (by^3 + 3a_2x^2y - a_3x^2z - b_3y^2z) \\ & - 4ba_2a_3n^3nz (- a_2{}^2n^2x^2 + ba_2n^2y^2 + a_3b_3n^2z^2) \\ =\; & ba_2{}^3n^6 (by^3 + 3a_2x^2y + 3a_3x^2z + 3b_3y^2z - 4a_3{}^2b_3z^3/a_2{}^2), \quad . \; . \quad (101) \end{aligned}$$

since, for the above form of u, with $\xi = 0$, $\eta = 0$, $\zeta = n$,

$$\mathrm{P} = - 2ba_2a_3n^3. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (102)$$

Now, the terms within brackets in ϖ are equal to

or
$$\left.\begin{aligned} (by - b_3z)(b^2y^2 + 3ba_2x^2 + 4bb_3yz + 4b_3{}^2z^2)/b^2 \\ (by - b_3z)(by + \sqrt{-3ba_2}x + 2b_3z)(by - \sqrt{-3ba_2}x + 2b_3z)/b^2, \end{aligned}\right\} \quad (103)$$

which factors may be identified with the three tangents to u at the points $u = 0$, $z = 0$; viz., the points

$$z = 0,\ y = 0\,; \qquad z = 0,\ y = \pm\sqrt{-3a_2/b}\,x.$$

These will be real points only when ba_2 are of unlike sign; *i.e.*, when L does not meet s in real points, and v is acnodal.

69. The Cotesian (98)—discarding the factor $4ba_2{}^3n^6$—

$$by^3 + 3a_2x^2y - a_3x^2z - b_3y^2z = 0$$

and the primitive cubic u (93) are plainly intersected by L, or $z = 0$, in the same three points, and the tangents to v at these points will be derivable from those to u by changing therein

$$3a_3 \text{ into } -a_3, \qquad 3b_3 \text{ into } -b_3\,;$$

viz., their equations are

$$\left.\begin{aligned} 3by + b_3z = 0, \\ \pm 3\sqrt{-3ba_2}\,x + 3by - 2b_3z = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

70. The equation to s (94) being

$$-ba_2x^2 + b^2y^2 - b_3{}^2z^2 = 0,$$

shows by its form that the tangent to u at the point

$$(\text{L or})\ z = 0, \qquad y = 0,$$

viz., the line (103)

$$by - b_3z = 0,$$

or

$$a_2y + a_3z = 0,$$

touches s at the point

$$x : y : z = 0 : b_3 : b.$$

71. Again, throwing the equation to s into the form

$$\begin{aligned} 3s &\equiv -(b^2y^2 + 3ba_2x^2 + 4bb_3yz + 4b_3{}^2z^2) + 4b^2y^2 - 4bb_3yz + b_3{}^2z^2, \\ &\equiv \{-3b\,(a_2x^2 + 3by^2) + 4\,(2by + b_3z)\,(by - b_3z)\} + (2by + b_3z)^2, \end{aligned}$$

the terms within brackets being (103) the tangents to u at the other two points at which L meets it, it appears that these lines touch s on the line

$$2by + b_3z = 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

i.e., at the points determined by it, and

$$a_2x^2 + 3by^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

the polars of those two points of the three common to L and u.

For these points are determined (93) by ($z = 0$) and

$$3a_2x^2 + by^2 = 0,$$

or

$$x : y : z = b : \pm\sqrt{-3ba_2} : 0,$$

the polars of which, with respect to (94)

$$a_2^2x^2 + a_2by^2 - a_3b_3z^2 = 0,$$

are

$$a_2bx \pm b\sqrt{-3ba_2}y = 0,$$

or

$$a_2x^2 + 3by^2 = 0.$$

72. The equation of the double chord (§ 41) of the pencil through $x'y'z'$ on L being generally (56)

$$\Sigma \frac{\partial s}{\partial x}\left(\frac{\partial s}{\partial z'}\frac{\partial u}{\partial y'} - \frac{\partial s}{\partial y'}\frac{\partial u}{\partial z'}\right) = 0,$$

is, in the present case, from the values of the differential coefficients given (98), § 65,

$$(a_2x'^2 - by'^2)(y'x - x'y) + (3b_3y'^2 - a_3x'^2)x'z = 0, \quad . \quad . \quad . \quad (107)$$

or, arranged as a binary cubic in $x'y'$,

$$-(a_2y + a_3z)x'^3 + a_2xx'^2y' + (by + 3b_3z)x'y'^2 - bxy'^3 = 0.$$

The discriminant of this last form gives the envelope of the double chord as the point $x'y'$ describes the line L, or $z = 0$; viz., it is

$$3w \equiv 4a_2^2x^2(2by - 3b_3z)^2 \\ - \{a_2^2x^2 + 3(a_2y + a_3z)(by + 3b_3z)\}\{3ba_2x^2 + (by + 3b_3z)^2\} \quad . \quad (108)$$

or

$$3w = -3ba_2(a_2x^2 + 3by^2)^2 \\ + 2a_2(a_2x^2 + 3by^2)(2by - 3b_3z)^2 - 3(2a_2y - a_3z)(2by - 3b_3z)^3, \quad (109)$$

in virtue of the relation (92)

$$ba_2 = -a_3b_3.$$

The former value of w shows that the polar of $y = 0$, $z = 0$, one of the three points common to L ($z = 0$) and u, viz.,

$$x = 0,$$

is the tangent at a cusp at the point

$$x = 0, \quad by + 3b_3z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

the tangent meeting w for the fourth time at the point

$$x = 0, \qquad a_2y + a_3z = 0, \qquad \text{or } by - b_3z = 0. \quad . \quad . \quad . \quad . \quad (111)$$

73. Now, since any real intersection of L and u might be taken as the point $y = 0$, $z = 0$, it follows that when these three intersections are real there will be three real cusps to w, the cuspidal tangents being the polars, with respect to the poloid s, of the points common to L and u.

This is shown, independently, for the two intersections of L and u other than $y = 0$, $z = 0$, by the second form of w, which gives simultaneously (109)

$$\left.\begin{aligned} (a_2x^2 + 3by^2)^2 = 0, \\ (2by + b_3z)(2by - 3b_3z)^3 = 0\,; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

of which the former has been shown to be the (square of the) s-polars of the two intersections in question (106).

74. It is plain from the forms of w that the triad of coordinates (110), (112) satisfy the first differential coefficients of w; in fact these are

$$\begin{aligned}
\frac{\partial w}{\partial x} &= -\,4a_2{}^2x\,\{2b\,(a_2x^2 + 3by^2) - (2by - 3b_3z)^2\}, \\
\frac{\partial w}{\partial y} &= 4ba_2\,\{a_2x^2\,(by - 6b_3z) - y\,(by + 3b_3z)^2\}, \\
&= 4ba_2\,\{(a_2x^2 + 3by^2)\,(by - 6b_3z) - y\,(2by - 3b_3z)^2\}, \\
\frac{\partial w}{\partial z} &= 4a_3\,\{3a_2bx^2 + (by + 3b_3z)^2\}\,(2by - 3b_3z), \\
&= 4a_3\,\{3b\,(a_2x^2 + 3by^2)\,(2by - 3b_3z) - (4by + 3b_3z)\,(2by - 3b_3z)^2\}.
\end{aligned}$$

Also

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= 4a_2{}^2\,\{-\,3b\,(a_2x^2 + by^2) + (2by - 3b_3z)^2\}, \\
\frac{\partial^2 w}{\partial x\,\partial y} &= 8a_2{}^2bx\,(by - 6b_3z), \\
\frac{\partial^2 w}{\partial x\,\partial z} &= -\,24a_2{}^2b_3x\,(2by - 3b_3)\,z,
\end{aligned}$$

giving for the cuspidal tangent

$$x\frac{\partial^2 w}{\partial x^2}+y\frac{\partial^2 w}{\partial x\,\partial y}+z\frac{\partial^2 w}{\partial x\,\partial z}=0,$$

when in $\frac{\partial^2 w}{\partial x^2}$ . . . are introduced the values $x=0$, $by=-3b_3z$,

$$x=0$$

as in (110); when the values $a_2{}^2x^2=-3by^2$, $2by=3b_3z$,

$$a_2x\pm\sqrt{-3a_2b}y=0$$

as in (112) above. The same values substituted in the second differential coefficients will be found to make

$$\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial z^2}-\left(\frac{\partial^2 w}{\partial y\,\partial z}\right)^2=0.$$

75. Considering the intersections of w and s, the elimination of x^2 between their equations leads to

$$(4b^3y^3-3bb_3{}^2yz^2-b_3{}^3z^3)\,z=0,$$

or

$$(by-b_3z)\,(2by+b_3z)^2\,z=0. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (113)$$

Referring to the equation of w (108), § 72, it appears at once that

$$a_2y+a_3z=0,$$

or (92)

$$by-b_3z=0,$$

gives

$$x^2=0,$$

viz., the tangent to u at the point $y=0$, $z=0$, (103), § 68, touches w at the point

$$x:y:z=0:b_3:b,$$

in which w meets s; and the same line has been shown (§ 70) to touch s at that point. From this it would follow at once that the other two tangents to u, at the points L or $z=0$, $u=0$, touch both w and s at the same points; but this is shown by (113) above, since w and s appear at once from it to have double contact at the points in which s is met by the line

$$2by+b_3z=0,$$

which has been proved (§ 71) to be the chord of contact with s of the two tangents referred to

$$3ba_2x^2 + (by + 2b_3z)^2 = 0.$$

76. Generally then, when the transversal L meets the cubic u in three real points, w the envelope of the double chords of pencils of lines through points in L, is a tricuspidal quartic having triple contact with s, the poloid of L, at the points in which it is touched by the three tangents to u at the points $L = 0$, $u = 0$; the cuspidal tangents being the s-polars of those three points; and the fourth pair of points common to w and s lie on the line L.

In Plate 8 the envelope w is figured for the case of L being the line at infinity and the cubic u a modification of the Cissoid, as more particularly described below (§ 91).

The property of the double chord touching w in a point which is harmonic to its second intersection with the poloid s relatively to its COTES-point and intersection with L, has been proved in § 59 by the use of other lines of reference, with the reality or imaginariness of which it is, plainly, unconnected.

77. When L touches u, it and the tangent at the point where it again meets the cubic being taken as

$$z = 0, \qquad y = 0,$$

and their chord of contact with the poloid s as

$$x = 0,$$

the equation of the poloid (10) is reduced, since $l = 0$, $m = 0$, to

$$(ab_1 - a_2^2)\,x^2 + (ba_3 + a_2b_3 - 2b_1e)\,yz = 0,$$

the conditions for which are (*ib.*)

$$ba_2 - b_1^2 = 0, \quad \dots\dots\dots\dots \quad \text{(i.)}$$

$$a_3b_3 - e^2 \quad = 0, \quad \dots\dots\dots\dots \quad \text{(ii.)}$$

$$ab_3 + a_3b_1 - 2a_2e = 0, \quad \dots\dots\dots\dots \quad \text{(iii.)}$$

$$ab - a_2b_1 = 0. \quad \dots\dots\dots\dots \quad \text{(iv.)}$$

But since $y = 0$, $z = 0$ is on u,

$$a = 0;$$

therefore, by (iv.),

$$b_1 = 0,$$

since $a_2 = 0$ is excluded by the form of s;

therefore, by (i.),

$$b = 0,$$

and by (iii.)

$$e = 0\,;$$

finally, by (ii.)

$$a_3 = 0,$$

since $b_3 = 0$ is excluded by the form of s.

Thus, it appears that the cubic u and poloid s are reduced to

$$u \equiv cz^3 + 3a_2x^2y + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y, \quad \ldots \ldots \quad (114)$$

$$s/n^2 \equiv - a_2{}^2x^2 + a_2b_2yz. \quad \ldots \ldots \ldots \ldots \quad (115)$$

The form of u shows that $x = 0$ passes through the point in which $z = 0$ touches it; or the poloid touches the cubic at the point of contact of the transversal L.

78. The coefficients of the reciprocal of s which do not vanish are

$$4\mathrm{A} = - a_2{}^2b_3{}^2n^4, \qquad 4\mathrm{F} = 2a_2{}^3b_3n^4\,;$$

which values, with $l = 0$, $m = 0$, give (37), § 30 (dividing out n^6),

$$\begin{aligned} v \equiv & - z^2\{- a_2{}^2b_3{}^2a_2y + 4a_2{}^3b_3(b_3y + c_2z)\} \\ & + 2z\{2a_2{}^3b(a_2x^2 + c_2z^2 + 2b_3yz)\} \\ & + 4(- a_2{}^2x^2 + a_2b_3yz)\,(- a_2{}^2b_3z) \\ = & \; a_2{}^3b_3z(8a_2x^2 + b_3yz)\,; \quad \ldots \ldots \ldots \ldots \quad (116) \end{aligned}$$

the locus degenerating in this case into the line L and a conic having double contact with s at the points where the s-polar of the point of intersection of L and u meets it and s.

79. The double chord through any point $x'y'$ on L or $z = 0$ (56),

$$\frac{\partial s}{\partial x}\left(\frac{\partial s'}{\partial z'}\frac{\partial u'}{\partial y'} - \frac{\partial s'}{\partial y'}\frac{\partial u'}{\partial z'}\right) + \ldots = 0$$

in this case, wherein

$$\frac{\partial s}{\partial x} : \frac{\partial s}{\partial y} : \frac{\partial s}{\partial z} = - 2a_2x : b_3z : b_3y,$$

$$\frac{\partial u'}{\partial x'} : \frac{\partial u'}{\partial y'} : \frac{\partial u'}{\partial z'} = 2a_2x'y' : a_2x'^2 : b_3y'^2,$$

is

$$- 2a_2x\,(\;*\; - a_2b_3x'^2y') + b_3z\,(2a_2b_3x'y'^2 + 2a_2b_3x'y'^2) + b_3y\,(- 2a_2{}^2x'^3 - \;*\;) = 0,$$

or

$$a_2x'y'x - a_2x'^2y + 2b_3y'^2z = 0, \quad \ldots \ldots \quad (117)$$

the envelope of which as $x'y'$ describes the line L or $z = 0$, is

$$w \equiv a_2x^2 + 8b_3yz = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

another conic having double contact with s and v at the same points.

80. Lastly, the chord through a point $(x''y''z'')$ on the polar line of $x'y'$, any point on L which has the former as its COTES-point, but which is not one of the pencil of chords through $(x'y'z')$, is (53)—when $z' = 0$, and L is $z = 0$—

$$\frac{\partial^2 u''}{\partial x''^2}\{y'z''(zx'' - xz'') + x'z''(zy'' - yz'')\} + 2\frac{\partial^2 u''}{\partial x''\partial y''}y'z''(zy'' - yz'') - 0.$$

But here

$$\frac{\partial^2 u''}{\partial x''^2} = 6a_2y'', \qquad \frac{\partial^2 u''}{\partial x''\partial y''} = 6a_2x'';$$

whence the chord in question is

$$-y'y''z''x - (x'y'' + 2y'x'')z''y + (x'y'' + 3y'x'')y''z = 0. \quad . \quad . \quad (119)$$

81. The polar line of $x'y'0$ being

$$2a_2x'y'x + a_2x'^2y + b_3y'^2z = 0,$$

the envelope of the chord (119) above, as $x''y''z''$ describes the polar line just mentioned, determined as the condition that

$$2a_2x'y'x'' + a_2x'^2y'' + b_3y'^2z'' = 0$$

shall touch

$$x'zy''^2 - (y'x + x'y)y''z'' - 2y'yz''x'' + 3y'zx''y'' = 0,$$

is—dividing out the factor y'^4—

$$(2a_2x'x + 3b_3y'z)^2 + 8a_2b_3x'^2yz = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

or, as it may be otherwise written,

$$(2a_2x'x - b_3y'z)^2 + 8b_3(2a_2x'y'x + a_2x'^2y + b_3y'^2z)z = 0, \quad . \quad . \quad (121)$$

a conic inscribed in the triangle formed by the lines $y = 0$, L or $z = 0$, and the polar line of the point $x'y'$ on L.

V. Newtonian Diameters.

82. If L is the line at infinity, then

$$l : m : n = \sin \mathrm{A} : \sin \mathrm{B} : \sin \mathrm{C},$$
$$= \alpha : \beta : \gamma ;$$

and to the Cotes-point of a chord through a given point on L $(x'y'z')$ corresponds the "mean point" on a chord parallel to a given line, the "direction coordinates" of which are, § 44,

$$\gamma\eta - \beta\zeta, \qquad \alpha\zeta - \gamma\xi, \qquad \beta\xi - \alpha\eta ; \quad . \quad . \quad . \quad . \quad . \quad (122)$$

the line being written

$$\xi x + \eta y + \zeta z = 0 ;$$

viz., the quantities (122) may be considered as the coordinates of the point in which this line, and all parallel to it, meet the line at infinity.

To every equation in the preceding part of this Paper involving as parameters $(x'y'z')$ the coordinates of a point on L, a finite line, now corresponds one in which $x'y'z'$ are replaced by

$$\lambda : \mu : \nu = \gamma\eta - \beta\zeta : \alpha\zeta - \gamma\xi - : \beta\xi - \alpha\eta. \quad . \quad . \quad . \quad (123)$$

Thus, for the polar line of $x'y'z'$ (7) now appears $\mathrm{D}^2 u$ (4), § 17, *i.e.*,

$$\lambda^2 \frac{\partial^2 u}{\partial x^2} + \mu^2 \frac{\partial^2 u}{\partial y^2} + \nu^2 \frac{\partial^2 u}{\partial z^2} + 2\mu\nu \frac{\partial^2 u}{\partial y\, \partial z} + 2\nu\lambda \frac{\partial^2 u}{\partial z\, \partial x} + 2\lambda\mu \frac{\partial^2 u}{\partial x\, \partial y} = 0, \quad . \quad (124)$$

which is, plainly, the locus of a point O on a chord parallel to $\xi x + \ldots = 0$, and meeting the cubic u in the points O_1, O_2, O_3, such that

$$\mathrm{OO}_1 + \mathrm{OO}_2 + \mathrm{OO}_3 = 0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

viz., O is the "mean point" on the chord relatively to the triad O_1, O_2, O_3.

The line (124) is the Newtonian Diameter of the system of chords $(\lambda\mu\nu)$; and its envelope, since (123)

$$\alpha\lambda + \beta\mu + \gamma\nu \equiv 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

the "centroid," *i.e.*, the "poloid" of the line at infinity, is

$$\alpha^2 \left\{ \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial y\, \partial z} \right)^2 \right\} + \ldots + 2\beta\gamma \left\{ \frac{\partial^2 u}{\partial z\, \partial x} \frac{\partial^2 u}{\partial x\, \partial y} - \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y\, \partial z} \right\} + \ldots = 0 \, . \quad (127)$$

or

$$u_{11} x^2 + \ldots + u_{23} yz + \ldots = 0,$$

where $u_{11} = 0$ is now the condition that $\partial_x u$ should be a parabola . . . ,

$u_{23} = 0$, that the line at infinity should be cut harmonically by the conics $\partial_y u$, $\partial_z u$ . . . .

The condition that the cubic u should touch the line at infinity in the standard form $(b^2c^2 + \ldots)\,\alpha^6 + \ldots = 0$ is equal to four times that for the centroid being parabolic and touching the finite asymptote of u; and

83. When the cubic meets the line at infinity in three real points, or the three asymptotes are real, the centroid is an ellipse inscribed in the triangle formed by these lines, so as to touch them at their middle points (Plate 8).

But if the cubic u has only one real intersection with the line at infinity, then the centroid is a hyperbola, having the single real asymptote as a tangent; and

The asymptotes of the hyperbolic centroid are the only *real pair of conjugate diameters of the cubic* u, viz., each cuts every chord parallel to the other in its mean point, and in particular divides the tangential chord of the cubic parallel to the other, or the parallel nodal chord if the cubic is nodal, in the ratio of 2 : 1. Thus in Plate 7 the chords BB_1, $B'B'_1$, B'_2, parallel to one asymptote of the centroid, are divided in such wise by the other asymptote in the points B_0, B'_0.

84. The mean point on any diameter of u regarded as a chord of that cubic is the point in which it is met by the diameter of the centroid conjugate in direction to its chords, the "double" one (§ 7) of which is the s-polar § 42 of that mean point.

The locus of the mean points of diameters of the cubic is the nodal cubic v, the Cotesian of the line at infinity, having as its nodal tangents the asymptotes of the centroid, and being therefore *acnodal* when the three asymptotes of the cubic are all real (Plate 8), the acnode, or conjugate point, being the centre of the elliptic centroid.

The asymptotes of the cubic v meeting, two and two, on the diameters of s through its points of triple contact with v, are parallel to those of the primitive cubic u; and the line at infinity is its inflexional axis, these inflexions being at the points in which the asymptotes of u meet that line.

In Plate 8 the Cotesian is represented with three hyperbolic branches, each touching the centroid; in Plate 7, with a real loop touching the centroid, and a single real asymptote parallel to that of the cubic u.

85. The envelope of the "double ordinates" of the Newtonian Diameters is represented in Plate 8 as a quartic (w) with three real cusps, the three diameters of the centroid conjugate to the asymptotes of u being the cuspidal tangents. This quartic w, as well as the Cotesian v, has triple contact with the centroid at the points of contact of the asymptotes of u. In Plate 7, the cubic u having only one real asymptote, the quartic w has only one real cusp; also it and v have only one real contact with s at the point of contact of the real asymptote with s. Within the limits of the diagram the only parts of w visible are (i.) that adjoining the contact referred to, terminating on one side with the point of contact (C′) of the double ordinate CD of the diameter DD_0—in this case (of L being at infinity) the point on CD conjugate to D (§ 59) being at infinity, D is the centre of the segment CC′—and (ii.) the cusp of w with the

contact C'_1 of another double ordinate $C'D'$, of the diameter $D'D'_0$, D' being the centre of $C'C'_1$. It will be observed that D_0, D'_0 are the mean points of the diameters DD_0, $D'D'_0$, D_0 being harmonic conjugate to D with respect to D_1D_2, the other two real intersections with v (§ 58).

86. In Plate (8) is shown an "alien" ordinate EE_0 of DD_0, having its mean point E_0 on that diameter (§ 61), but being an ordinate of the other tangent to s which might be drawn through E_0. The parabola, which is the envelope of these "alien" ordinates through the different points on DD_0, is also shown, touching DD_0 at the point D_0, and having the connector of that point with the centre of the centroid as diameter (88), this connector being now the representative of the

$$y'x + x'y = 0$$

of the equation just cited, the connector of the COTES-point on the polar line of $(x'y'0)$ with the pole of the transversal L, now become the line at infinity.

87. The asymptotes of the centroid offer themselves as an unique pair of Cotesian axes to which the cubic may be frequently referred with advantage, its equation being then

$$u \equiv ax^3 + by^3 + 6exy + 3c_1x + 3c_2y + c = 0, \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

and the centroid, with changed sign—which is now immaterial—

$$s \equiv abxy - e^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

For many discussions also it is convenient to define the diameter by the coordinates $(x''y'')$ of its point of contact with s.

The equation of its double ordinate will now be

$$ax''(x - x'') + e(y - y'') = 0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

and that of its "alien" ordinate at the point x_1y_1

$$ax_1y''(x - x_1) + ey_1(y - y_1) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

It will be sufficient to indicate the steps by which these may be verified independently of the general formulæ given in the earlier part of this Memoir.

The diameter is the tangent to (ii.) at $x''y''$ or

$$y''x + x''y - 2x''y'' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

the coordinates of the mean point of which may be found as one-third of the sums of those of its intersections with u. The polar of this mean point is (iii.); and that of the mean point of the second tangent to s through x_1y_1 on (v.) is (iv.)

88. The secondary chords through points lying on a given diameter, and having them as their mean points, being inclined at varying angles with that diameter, to determine that one which makes a given angle—say θ—with the diameter

$$\tan\theta = y''(ax''x_1 - ey_1)\sin\omega/\{ay''^2x_1 + ex'y_1 - y''(ax''x_1 + ey_1)\cos\omega\},$$

ω being the angle between the asymptotes of the centroid: viz., it is the chord which meets the diameter in the point (x_1y_1) such that

$$x_1 : y_1 = e\{y''(\cos\omega - \sin\omega) + x''\tan\theta\} : ax''\{x''(\cos\omega + \sin\omega) - y''\tan\theta\}.$$

In particular, the point at which the secondary chord is perpendicular to the diameter is determined by

$$x_1 : y_1 = e\,(x'' + y''\cos\omega) : -\,ay''(y'' + x''\cos\omega).$$

The angle between a diameter and its primary chords or ordinates is given by

$$\tan\phi = (ax''^2 - ey'')\sin\omega/\{x''(ay'' + e) - (ax''^2 + ey'')\cos\omega\},$$

and those diameters which are perpendicular* to their chords are the tangents to the centroid at the points determined by

$$x\,(ay + e) = (ax^2 + ey)\cos\omega,$$

i.e., the intersections of the conic

$$a\,(x^2\cos\omega - xy) + e\,(-\,x + y\cos\omega) = 0$$

with the centroid; or, multiplying by b, those of the parabolas

$$abx^2\cos\omega + be\,(-\,x + y\cos\omega) - e^2 = 0,$$
$$aby^2\cos\omega + ae\,(x\cos\omega - y) - e^2 = 0.$$

The finite intersections in question are only three in number, their coordinates being determined by the cubics

$$a^2bx^3\cos\omega - abex^2 - ae^2x + e^3\cos\omega = 0,$$
$$ab^2y^3\cos\omega - abey^2 - be^2y + e^3\cos\omega = 0.$$

89. When the asymptotes of the centroid are imaginary, two convenient Cotesian axes of coordinates are found in the diameter parallel to one asymptote of u and its

---

* "Diameter autem ad Ordinatas rectangula si modo aliqua sit, etiam Axis dici potest." NEWTON, 'Enumeratio Linearum Tertii Ordinis,' § 2.

conjugate; to which lines the other two asymptotes will form a conjugate harmonic pair. These axes are the special form which those of the self-conjugate triangle of reference employed, §§ 63–76, take; and all the equations there deduced will be adapted to the above Cotesian axes simply by substituting unity for z.

90. When the cubic u is parabolic, *i.e.*, touches the line at infinity, its "centroid" s also touches that line, viz., it becomes a parabola, which may then be referred to the tangent parallel to the finite asymptote of u, and the diameter through its point of contact.

The Cotesian v will then be made up of the line at infinity and a parabola, and the chords of u having their mean points on a diameter of which they are *not* ordinates will envelope another parabola, while the double chord will envelope a third.

In this case all the equations and formulæ of §§ 77–81 are applicable by simply making $z = 1$.

It has not been thought necessary to add a figure in illustration of this case, the curves being all of familiar character.

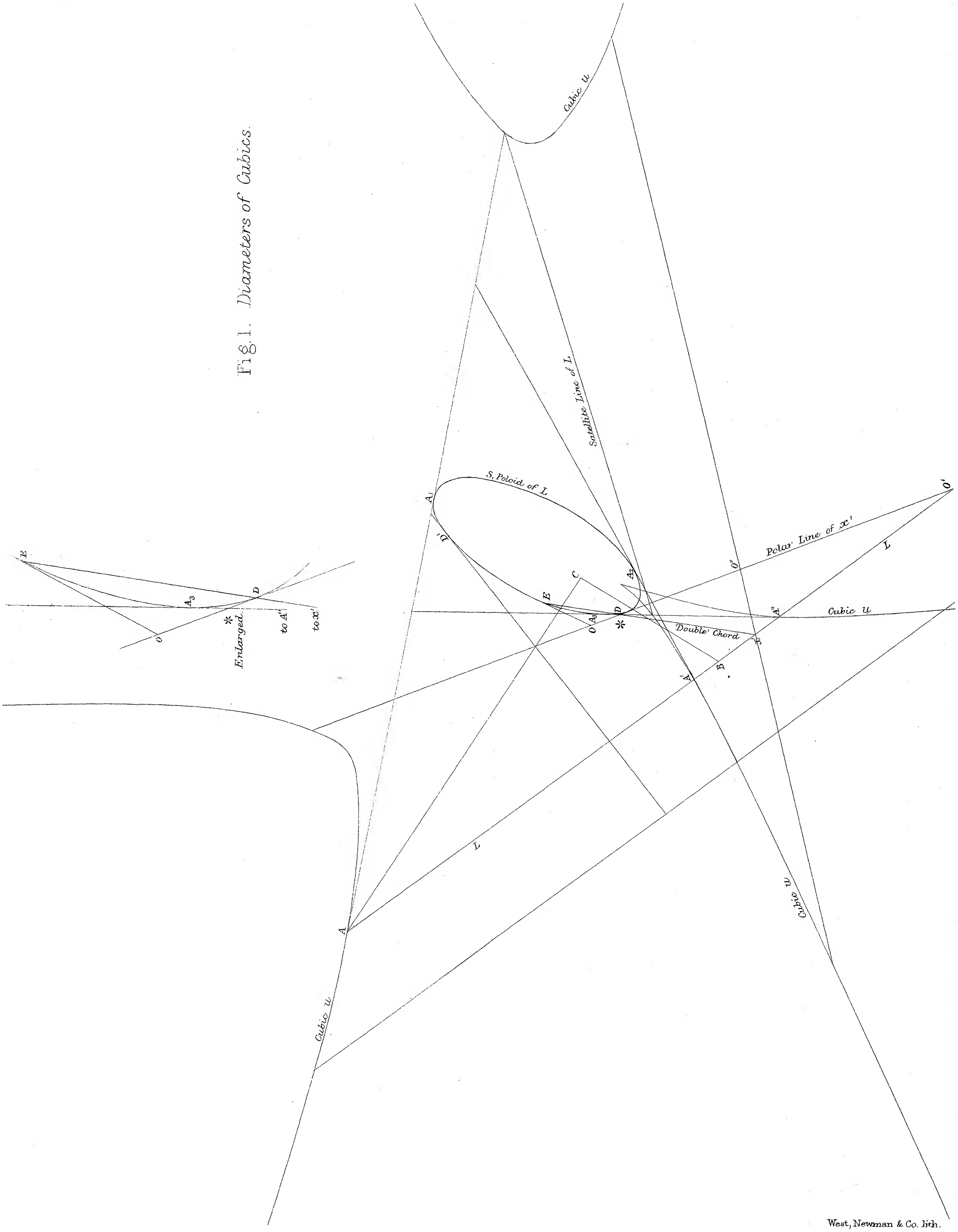
VI.—Description of Plates.

91. The two types of cubic which have been drawn to illustrate results arrived at in this Memoir have been constructed geometrically with great accuracy as follows:—

(i.) Those in Plate 6 and Plate 8 from two conjugate diameters of a hyperbola, as the well-known Cissoid of Diocles from two rectangular diameters of a circle; viz., from a vertex of one diameter a pencil of lines was drawn, each to the extremity of an ordinate parallel to the other diameter, and then its intersection with the equidistant ordinate on the other side of that diameter determined a point the locus of which gave the cubic u as represented, with one cuspidal branch and two hyperbolic. For the figure in Plate 6 the line L was drawn arbitrarily, cutting u in three real points A, A′, A″, and the tangents to u at these points traced, the accuracy of their directions being vouched by their tangential points proving to range in a right line KK_1K_2. The poloid s was next inscribed in the triangle formed by the triad of tangents, touching them at points A_1, A_2, A_3, harmonic-conjugate severally to A, A′, A″ with respect to the corners of the triangle. Another arbitrary line OD having been drawn touching the conic in D and meeting the cubic in three real points, its Cotes-point O was found by actual calculation from the measured lengths of the segments between L and the cubic u. The polar of O, being the double chord of the pencil through the point on L of which OD was the polar line, of course determined that point (x'). [Otherwise x' might have been taken arbitrarily and the Cotes-point on L relatively to it have been found by measurement; and so the polar-line of x' with the double chord have been arrived at.] The accuracy of the figure so far was tested by examining the agreement of the Cotes-points of L itself and another chord with their intersections by the polar line at O′, O″. For the figure in Plate 8, correspondingly, the "centroid" was inscribed in the triangle formed by the asymptotes of *u* so as to bisect the sides. The Cotesian *v* and the quartic *w* were laid down from their equations without much difficulty, their triple contacts with *s*, and the parallelism of the asymptotes of *v* to those of *u*, combined with the simply defined positions of the cusps, and directions of the cuspidal tangents of *w*, enabling them to be traced with great accuracy. The parabolic envelope shown in connexion with the diameter DD_0, touching that line at its mean point and having an axis parallel to the connector of that point with the centre of the centroid, as well as touching the asymptotes, was readily constructed geometrically from these data, and its accuracy tested by a tangent drawn arbitrarily, meeting *u* in three real points E, E_1, E_2 and cutting DD_0 in E_0, proving to have the last as its mean point, as determined by actual measurement of segments, with great exactness.

(ii.) For the second illustration, Plate 7, the figure of a cubic *u* having only a single real asymptote was obtained by constructing a Cissoid from two conjugate diameters of an ellipse, precisely as that of DIOCLES from the circle, of which in fact this, as well as the figure previously described, may be regarded as projections. The hyperbolic centroid was then constructed from its equation referred to the two conjugate diameters of the generating ellipse, which are also conjugate diameters of the centroid, and to one of which the real asymptote of *u* is parallel. The Cotesian *v* with its real loop and single real asymptote, parallel to that of *u*, having also the asymptotes of the centroid as nodal tangents (§ 50), was constructed from its equation, the loop touching the centroid at the contact with it of the real asymptote of *u*. It will be remarked in this as in the case preceding where they are all three real, how soon the cubic *v* approaches its asymptote, and its curvature becomes inappreciable. As regards the little of *w* visible within the limits of the figure some remarks have already been made, § 60. Two diameters, besides the conjugate pair which the asymptotes of *s* form, § 83, have been introduced, DD_0 touching one branch of *s* and having three real intersections with *v*; $D'D'_0$ touching the other branch of *s* and meeting *v* in only one, its mean point, D'_0. The "double" chords of these diameters, polars of their mean points, touching their envelope *w* at the points C_1, C'_1, equidistant from D, D′ respectively as C, C′, are typical of their kind. It will be remembered that they are "double" chords of *u*, with which, however, in this figure, they have only one real intersection. The mean point of the chord AA′A″, drawn parallel to the double chord CD, as determined by actual measurement of segments, coincides with A_0, its intersection with the diameter DD_0, to a nicety. Some remarks have already been made, § 83, on the chords BB_1, $B'B'_1B'_2$ parallel to one asymptote of *s* and cut by the other in their mean points B_0, B_0' with great exactness. It has not been thought desirable to introduce the parabolic envelope connected with either of the diameters drawn, into this figure.

Fig. 1. Diameters of Cubics.



West, Newman & Co. lith.

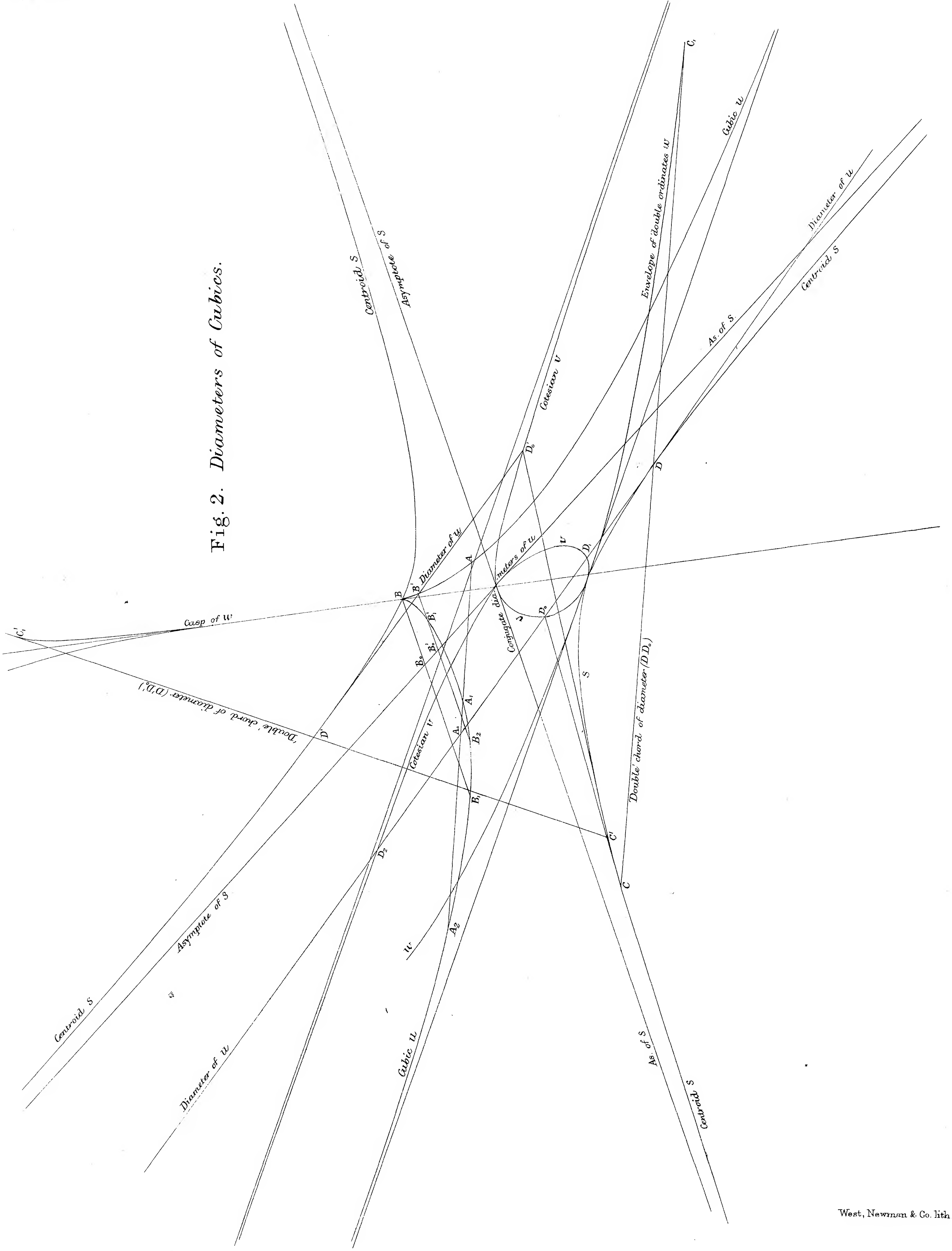
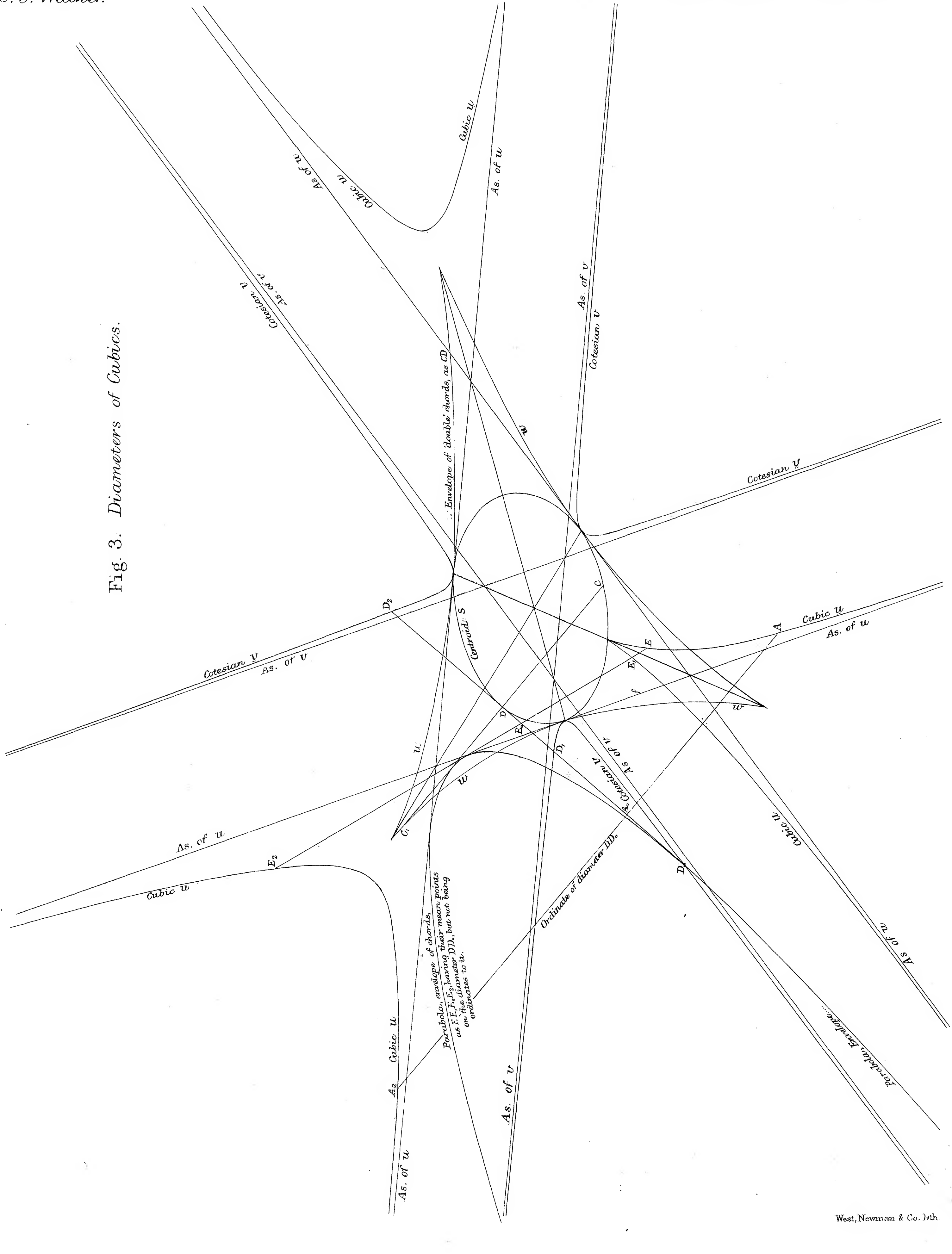


Fig. 2. Diameters of Cubics.

West, Newman & Co. lith.

Fig. 3. Diameters of Cubics.



West, Newman & Co. lith.